PARCIJALNI IZVODI I DIFERENCIJALI

Sama definicija parcijalnog izvoda i diferencijala je malo teža , mi se njome ovde nećemo baviti a vi ćete je , naravno, naučiti onako kako vaš profesor zahteva. Mi ćemo probati da vas naučimo kako se konkretno traže parcijalni izvodi...

Najpre par reči o obeležavanjima: najčešće se u zadacima zadaje funkcija $z = z(x, y)$, pa je:

$\frac{\partial z}{\partial x} \rightarrow$ oznaka za parcijalni izvod " po x-su"

$\frac{\partial z}{\partial y} \rightarrow$ oznaka za parcijalni izvod " po y"

$\frac{\partial^2 z}{\partial x^2} \rightarrow$ oznaka za dupli parcijalni izvod " po x-su" , a računa se $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$

$\frac{\partial^2 z}{\partial y^2} \rightarrow$ oznaka za dupli parcijalni izvod " po y" , a računa se $\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$

$\frac{\partial^2 z}{\partial y \partial x}$ i $\frac{\partial^2 z}{\partial x \partial y}$ su mešoviti dupli parcijalni izvodi a traže se $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$ i $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$

Ovde je **najvažnije** zapamtiti sledeću stvar:

$\rightarrow$ **Kad tražimo parcijalni izvod " po x-su" tada *y* tretiramo kao konstantu (broj)**

$\rightarrow$ **Kad tražimo parcijalni izvod " po y " tada *x* tretiramo kao konstantu (broj)**

$\boxed{\text{primer 1.}}$

Odrediti prve parcijalne izvode za funkciju $z = x^2 + y^2 - 2x + 3y$

<u>Rešenje:</u>

Prvo tražimo $\frac{\partial z}{\partial x} \rightarrow$ parcijalni izvod po x. Znači da je y konstanta. $z = x^2 + \boxed{y^2} - 2x + \boxed{3y}$

Znamo da je izvod od konstante 0 kad nije vezana za funkciju , pa je:

$$z=x^2+y^2-2x+3y$$
$$\frac{\partial z}{\partial x}=2x+0-2+0=\boxed{2x-2}$$

Sad tražimo po y:

$z=\boxed{x^2}+y^2-\boxed{2x}+3y$ zaokruženo su konstante, pa je izvod od njih 0.

$$z=x^2+y^2-2x+3y$$
$$\frac{\partial z}{\partial y}=0+2y-0+3=\boxed{2y+3}$$

$\boxed{\text{primer 2.}}$

Odrediti prve parcijalne izvode za funkciju $z=3x^3y-6xy+5y^2+7x-12y$

<u>Rešenje:</u>

$$z=3x^3y-6xy+5y^2+7x-12y$$
$$\frac{\partial z}{\partial x}=3y\cdot 3x^2-6y\cdot 1+0+7-0=\boxed{9x^2y-6y+7}$$

Sad su konstante vezane za funkciju, njih prepišemo a tražimo normalno izvod od funkcije po x, recimo za

$3x^3y$ konstanta je 3y koje prepisujemo a izvod od x^3 je $3x^2$.

Da nađemo prvi parcijalni izvod po y:

$$z=3x^3y-6xy+5y^2+7x-12y$$
$$\frac{\partial z}{\partial y}=3x^3\cdot 1-6x\cdot 1+10y+0-12=\boxed{3x^3-6x+10y-12}$$

Kad radimo po *y*, sve što ima x je konstanta, pa je tako recimo za $3x^3y$ izraz $3x^3$ konstanta koju prepisujemo a

znamo da je od *y* izvod 1.

primer 3.

Odrediti prve parcijalne izvode za funkciju $z = \frac{3x}{y} + \frac{7y}{x}$

Rešenje:

$$z = \frac{3x}{y} + \frac{7y}{x}$$

$$\frac{\partial z}{\partial x} = \frac{3}{y} \cdot 1 + 7y \cdot (-\frac{1}{x^2}) = \boxed{\frac{3}{y} - \frac{7y}{x^2}}$$

$$\frac{\partial z}{\partial y} = 3x \cdot (-\frac{1}{y^2}) + \frac{7}{x} \cdot 1 = \boxed{-\frac{3x}{y^2} + \frac{7}{x}}$$

primer 4.

Nađi prve parcijalne izvode za funkciju $u = \ln(x + y^2)$

Rešenje:

Pazite, ovde imamo i izvod složene funkcije:

$$u = \ln(x + y^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x + y^2} \cdot (x + y^2)`_{\text{po x}} = \frac{1}{x + y^2} \cdot (1 + 0) = \boxed{\frac{1}{x + y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x + y^2} \cdot (x + y^2)`_{\text{po y}} = \frac{1}{x + y^2} \cdot (0 + 2y) = \boxed{\frac{2y}{x + y^2}}$$

primer 5.

Nađi prve parcijalne izvode za funkciju $z = x^y$

Rešenje:

$$z = x^y$$

$\frac{\partial z}{\partial x} = y \cdot x^{y-1}$ ovde radimo kao $(x^{\Theta})` = \Theta \cdot x^{\Theta - 1}$

$$\frac{\partial z}{\partial y} = x^y \cdot \ln x$$

Za parcijalni izvod po y radimo kao : $(a^y)` = a^y \ln a$

primer 6.

Odrediti prve parcijalne izvode za funkciju $z=\frac{x+y}{x^2+y^2}$

Rešenje:

Ovde moramo raditi kao izvod količnika:

$$z=\frac{x+y}{x^2+y^2}$$

$$\frac{\partial z}{\partial x}=\frac{(x+y)`_{\text{po x}}\cdot(x^2+y^2)+(x^2+y^2)`_{\text{po x}}\cdot(x+y)}{(x^2+y^2)^2}$$

$$\frac{\partial z}{\partial x}=\frac{1\cdot(x^2+y^2)+2x\cdot(x+y)}{(x^2+y^2)^2}=\frac{x^2+y^2+2x^2+2xy}{(x^2+y^2)^2}=\boxed{\frac{3x^2+y^2+2xy}{(x^2+y^2)^2}}$$

$$\frac{\partial z}{\partial y}=\frac{(x+y)`_{\text{po y}}\cdot(x^2+y^2)+(x^2+y^2)`_{\text{po y}}\cdot(x+y)}{(x^2+y^2)^2}$$

$$\frac{\partial z}{\partial y}=\frac{1\cdot(x^2+y^2)+2y\cdot(x+y)}{(x^2+y^2)^2}=\frac{x^2+y^2+2xy+2y^2}{(x^2+y^2)^2}=\boxed{\frac{x^2+3y^2+2xy}{(x^2+y^2)^2}}$$

primer 7.

Odrediti prve parcijalne izvode za funkciju $u=\ln(x^2+y^2+z^2)$

Rešenje:

$$u=\ln(x^2+y^2+z^2)$$

$$\frac{\partial u}{\partial x}=\frac{1}{x^2+y^2+z^2}\cdot(x^2+y^2+z^2)`_{\text{po x}}=\frac{1}{x^2+y^2+z^2}\cdot 2x=\boxed{\frac{2x}{x^2+y^2+z^2}}$$

$$\frac{\partial u}{\partial y}=\frac{1}{x^2+y^2+z^2}\cdot(x^2+y^2+z^2)`_{\text{po y}}=\frac{1}{x^2+y^2+z^2}\cdot 2y=\boxed{\frac{2y}{x^2+y^2+z^2}}$$

$$\frac{\partial u}{\partial z}=\frac{1}{x^2+y^2+z^2}\cdot(x^2+y^2+z^2)`_{\text{po z}}=\frac{1}{x^2+y^2+z^2}\cdot 2z=\boxed{\frac{2z}{x^2+y^2+z^2}}$$

$\boxed{\text{primer 8.}}$

Odrediti prve parcijalne izvode za funkciju $u = \left(\frac{x}{y}\right)^z$

<u>Rešenje:</u>

$$u = \left(\frac{x}{y}\right)^z$$

$$\frac{\partial u}{\partial x} = z\left(\frac{x}{y}\right)^{z-1} \cdot \left(\frac{x}{y}\right)^{`}_{\text{po x}} = z\left(\frac{x}{y}\right)^{z-1} \cdot \frac{1}{y} = \boxed{\frac{z}{y} \cdot \left(\frac{x}{y}\right)^{z-1}}$$

$$\frac{\partial u}{\partial y} = z\left(\frac{x}{y}\right)^{z-1} \cdot \left(\frac{x}{y}\right)^{`}_{\text{po y}} = z\left(\frac{x}{y}\right)^{z-1} \cdot x \cdot \left(-\frac{1}{y^2}\right) = \boxed{-\frac{z}{y} \cdot \left(\frac{x}{y}\right)^{z}}$$

$$\frac{\partial u}{\partial z} = \boxed{\left(\frac{x}{y}\right)^z \cdot \ln\left(\frac{x}{y}\right)}$$

$\boxed{\text{primer 9.}}$

Odrediti $\frac{\partial^2 z}{\partial x^2} = ?, \quad \frac{\partial^2 z}{\partial x \partial y} = ?, \quad \frac{\partial^2 z}{\partial y \partial x} = ?, \quad \frac{\partial^2 z}{\partial y^2} = ?$ **za funkciju** $z = 3x^3 y - 6xy + 5y^2 + 7x - 12y$

<u>Rešenje:</u>

Naravno, najpre moramo naći prve parcijalne izvode:

$$z = 3x^3 y - 6xy + 5y^2 + 7x - 12y$$

$$\frac{\partial z}{\partial x} = 3y \cdot 3x^2 - 6y \cdot 1 + 0 + 7 - 0 = \boxed{9x^2 y - 6y + 7}$$

$$\frac{\partial z}{\partial y} = 3x^3 \cdot 1 - 6x \cdot 1 + 10y + 0 - 12 = \boxed{3x^3 - 6x + 10y - 12}$$

Sad koristeći njih tražimo dalje:

$$\frac{\partial z}{\partial x} = \boxed{9x^2y - 6y + 7}$$

$$\frac{\partial z}{\partial y} = \boxed{3x^3 - 6x + 10y - 12}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(9x^2y - 6y + 7) = \boxed{18xy}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(9x^2y - 6y + 7) = \boxed{9x^2 - 6}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(3x^3 - 6x + 10y - 12) = \boxed{9x^2 - 6}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(3x^3 - 6x + 10y - 12) = \boxed{10}$$

primer 10.

Odrediti $\frac{\partial^2 z}{\partial x^2} = ?, \quad \frac{\partial^2 z}{\partial x \partial y} = ?, \quad \frac{\partial^2 z}{\partial y \partial x} = ?, \quad \frac{\partial^2 z}{\partial y^2} = ?$ **za funkciju** $z = e^{xy}$

<u>Rešenje:</u>

$$z = e^{xy}$$

$$\frac{\partial z}{\partial x} = e^{xy} \cdot (xy)`_{\text{po x}} = e^{xy} \cdot y = ye^{xy} \rightarrow \boxed{\frac{\partial z}{\partial x} = ye^{xy}}$$

$$\frac{\partial z}{\partial y} = e^{xy} \cdot (xy)`_{\text{po y}} = e^{xy} \cdot x = xe^{xy} \rightarrow \boxed{\frac{\partial z}{\partial y} = xe^{xy}}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(ye^{xy}) = y \cdot ye^{xy} = \boxed{y^2e^{xy}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(ye^{xy}) = 1 \cdot e^{xy} + xe^{xy} \cdot y = \boxed{e^{xy}(1 + xy)}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(xe^{xy}) = 1 \cdot e^{xy} + ye^{xy} \cdot x = \boxed{e^{xy}(1 + xy)}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(xe^{xy}) = x \cdot xe^{xy} = \boxed{x^2e^{xy}}$$

$\boxed{\text{primer 11.}}$

Pokazati da je $\frac{x}{y}\frac{\partial z}{\partial x}+\frac{1}{\ln x}\frac{\partial z}{\partial y}=2z$ **ako je** $z=x^y$

<u>Rešenje:</u>

Kod ovog tipa zadatka najpre nađemo parcijalne izvode koji se javljaju u zadatku (ovde na levoj strani jednakosti)

Zamenimo ih i sredimo da dobijemo desnu stranu:

$$z=x^y$$
$$\frac{\partial z}{\partial x}=y\cdot x^{y-1}$$
$$\frac{\partial z}{\partial y}=x^y\cdot\ln x$$

Sad prepišemo levu stranu i zamenimo parcijalne izvode:

$$\frac{x}{y}\frac{\partial z}{\partial x}+\frac{1}{\ln x}\frac{\partial z}{\partial y}=$$
$$\frac{x}{\cancel{y}}\cdot\cancel{y}x^{y-1}+\frac{1}{\cancel{\ln x}}\cdot x^y\cancel{\ln x}=$$
$$x^y+x^y=2x^y=2z$$

Dokazali smo traženu jednakost.

$\boxed{\text{primer 12.}}$

Pokazati da je $x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=x$ **ako je** $z=x+\varphi(xy)$

<u>Rešenje:</u>

$$z=x+\varphi(xy)$$

$$\frac{\partial z}{\partial x}=1+\varphi`(xy)\cdot(xy)`_{\text{po x}}=1+\varphi`(xy)\cdot y=\boxed{1+y\cdot\varphi`(xy)}$$

$$\frac{\partial z}{\partial y}=0+\varphi`(xy)\cdot(xy)`_{\text{po y}}=\varphi`(xy)\cdot x=\boxed{x\cdot\varphi`(xy)}$$

zamenimo u levoj strani:

$$x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=$$

$$x\cdot(1+y\cdot\varphi`(xy))-y\cdot x\cdot\varphi`(xy)=$$

$$x+\cancel{xy\varphi`(xy)}-\cancel{xy\varphi`(xy)}=\boxed{x}$$

Dobili smo desnu stranu jednakosti!

$\boxed{\text{primer 13.}}$

Pokazati da je $y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=0$ **ako je** $z=\varphi(\sqrt{x^2+y^2})$

Rešenje:

$$z=\varphi(\sqrt{x^2+y^2})$$

$$\frac{\partial z}{\partial x}=\varphi`(\sqrt{x^2+y^2})\cdot(\sqrt{x^2+y^2})`_{\text{po x}}=\varphi`(\sqrt{x^2+y^2})\cdot\frac{1}{2\sqrt{x^2+y^2}}(x^2+y^2)`_{\text{po x}}=\varphi`(\sqrt{x^2+y^2})\cdot\frac{1}{\cancel{2}\sqrt{x^2+y^2}}\cdot\cancel{2}x$$

$$\boxed{\frac{\partial z}{\partial x}=\varphi`(\sqrt{x^2+y^2})\cdot\frac{x}{\sqrt{x^2+y^2}}}$$

$$\frac{\partial z}{\partial y}=\varphi`(\sqrt{x^2+y^2})\cdot(\sqrt{x^2+y^2})`_{\text{po y}}=\varphi`(\sqrt{x^2+y^2})\cdot\frac{1}{2\sqrt{x^2+y^2}}(x^2+y^2)`_{\text{po y}}=\varphi`(\sqrt{x^2+y^2})\cdot\frac{1}{\cancel{2}\sqrt{x^2+y^2}}\cdot\cancel{2}y$$

$$\boxed{\frac{\partial z}{\partial y}=\varphi`(\sqrt{x^2+y^2})\cdot\frac{y}{\sqrt{x^2+y^2}}}$$

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=$$

$$y\cdot\varphi`(\sqrt{x^2+y^2})\cdot\frac{x}{\sqrt{x^2+y^2}}-x\cdot\varphi`(\sqrt{x^2+y^2})\cdot\frac{x}{\sqrt{x^2+y^2}}=$$

$$\cancel{\varphi`(\sqrt{x^2+y^2})\cdot\frac{xy}{\sqrt{x^2+y^2}}}-\cancel{\varphi`(\sqrt{x^2+y^2})\cdot\frac{xy}{\sqrt{x^2+y^2}}}=0$$

TOTALNI DIFERENCIJAL funkcije $z = z(x, y)$ u oznaci dz se traži po formuli:

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Dakle, nađemo parcijalne izvode i zamenimo ih u formulu.

primer 14.

Naći totalni diferencijal sledećih funkcija:

a) $z = x^2 y$

b) $u = \dfrac{z}{x^2 + y^2}$

Rešenje:

a)

$$z = x^2 y$$

$$\frac{\partial z}{\partial x} = 2xy$$

$$\frac{\partial z}{\partial y} = x^2$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$\boxed{dz = 2xydx + x^2 dy}$$

b)

$$u = \frac{z}{x^2 + y^2}$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$\frac{\partial u}{\partial x} = z \cdot (-\frac{1}{(x^2 + y^2)^2}) \cdot 2x = -\frac{2xz}{(x^2 + y^2)^2} \to \boxed{\frac{\partial u}{\partial x} = -\frac{2xz}{(x^2 + y^2)^2}}$$

$$\frac{\partial u}{\partial y} = z \cdot (-\frac{1}{(x^2 + y^2)^2}) \cdot 2y = -\frac{2yz}{(x^2 + y^2)^2} \to \boxed{\frac{\partial u}{\partial y} = -\frac{2yz}{(x^2 + y^2)^2}}$$

$$\boxed{\frac{\partial u}{\partial z} = \frac{1}{x^2 + y^2}}$$

Sad ovo zamenimo u formulu:

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$$

$$\boxed{du=-\frac{2xz}{(x^2+y^2)^2}dx-\frac{2yz}{(x^2+y^2)^2}dy+\frac{1}{x^2+y^2}dz}$$

Ako u zadacima traže totalni diferencijal višeg reda , onda radimo, na primer:

Za $u=u(x,y)$ je $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$ a totalni diferencijali drugog i trećeg reda bi bili:

$$d^2u=(\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy)^2=\frac{\partial^2 u}{\partial x^2}dx^2+2\frac{\partial^2 u}{\partial x\partial y}dxdy+\frac{\partial^2 u}{\partial y^2}dy^2$$

$$d^3u=(\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy)^3=\frac{\partial^3 u}{\partial x^3}dx^3+3\frac{\partial^3 u}{\partial x^2\partial y}dx^2dy+3\frac{\partial^3 u}{\partial x\partial y^2}dxdy^2+\frac{\partial^3 u}{\partial y^3}dy^3$$

primer 15.

Ako je $u=x^3+y^3-3x^2y+3xy^2$ **nađi** d^2u

Rešenje:

$$u=x^3+y^3-3x^2y+3xy^2$$
$$\frac{\partial u}{\partial x}=3x^2-6xy+3y^2\rightarrow\frac{\partial^2 u}{\partial x^2}=6x-6y$$
$$\frac{\partial u}{\partial y}=3y^2-3x^2+6xy\rightarrow\frac{\partial^2 u}{\partial y^2}=6y+6x$$
$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}=-6x+6y$$

$$d^2u=\frac{\partial^2 u}{\partial x^2}dx^2+2\frac{\partial^2 u}{\partial x\partial y}dxdy+\frac{\partial^2 u}{\partial y^2}dy^2$$
$$d^2u=(6x-6y)dx^2+2(-6x+6y)dxdy+(6y+6x)dy^2$$