VII) Израчунати интеграл:

1. $\int \frac{dx}{\mathrm{sh}x} = \int \frac{\mathrm{ch}^2x - \mathrm{sh}^2x}{\mathrm{sh}x}dx = \int \frac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{4\cdot 2} \cdot \frac{2\cdot 1}{e^x-e^{-x}}dx =$

$= \frac{1}{2}\int \frac{e^{2x}+2+e^{-2x}-e^{2x}+2-e^{-2x}}{e^x-e^{-x}}dx = 2\int \frac{e^x dx}{e^{2x}-1} = 2\int \frac{da}{a^2-1} =$

$e^x = a$
$e^x dx = da$

$= 2\ln\left|\frac{e^x-1}{e^x+1}\right| + c$

2. $\int \frac{dx}{\mathrm{ch}x} = \int \frac{2dx}{e^x+e^{-x}} = \int \frac{2e^x dx}{e^{2x}+1} = 2\int \frac{da}{a^2+1} = 2\,\mathrm{arctg}(e^x) + c$

$e^x = a$
$e^x dx = da$

3. $\int \frac{dx}{\mathrm{sh}x\,\mathrm{ch}^3x} = \int \frac{\mathrm{ch}^2x - \mathrm{sh}^2x}{\mathrm{sh}x\,\mathrm{ch}^3x}dx = \int \frac{dx}{\mathrm{sh}x\,\mathrm{ch}x} - \int \frac{\mathrm{sh}x}{\mathrm{ch}^3x}dx =$

$= \int \frac{2\cdot 2\,dx}{(e^x+e^{-x})\cdot(e^x-e^{-x})} - \int a^{-3}da = 4\int \frac{dx}{e^{2x}-e^{-2x}} + \frac{1}{2\,\mathrm{ch}^2x} =$

$\mathrm{ch}x = a$
$\mathrm{sh}x\,dx = da$
$2x = b$
$2dx = db$
$dx = db/2$
$e^b = c$
$e^b db = dc$

$= 2\int \frac{db}{e^b-e^{-b}} + \frac{1}{2\,\mathrm{ch}^2x} = 2\int \frac{e^b db}{e^{2b}-1} + \frac{1}{2\,\mathrm{ch}^2x} =$

$= 2\int \frac{dc}{c^2-1} + \frac{1}{2\,\mathrm{ch}^2x} = 2\ln\left|\frac{e^{2x}-1}{e^{2x}+1}\right| + \frac{1}{2\,\mathrm{ch}^2x} + c$

4. $\int \frac{dx}{1+10\,\mathrm{ch}x} = \int \frac{dx}{1+5(e^x+e^{-x})} = \int \frac{e^x dx}{e^x+5e^{2x}+5} = \int \frac{da}{5a^2+a+5} = \frac{1}{5}\int \frac{da}{a^2+\frac{1}{5}a+1} =$

$e^x = a$
$e^x dx = da$

$= \frac{1}{5}\int \frac{da}{(a+\frac{1}{10})^2+\frac{99}{100}} = \frac{1}{5}\cdot\frac{100}{99}\int \frac{da}{\left(\frac{a+1110}{3\sqrt{111}10}\right)^2+1} = \frac{20}{99}\cdot\frac{3\sqrt{11}}{10}\int \frac{db}{b^2+1} =$

$\frac{a+1110}{3\sqrt{111}10} = b$
$\frac{da}{3\sqrt{111}10} = db$
$da = 3\sqrt{111}10\,db$

$= \frac{2\sqrt{11}}{33}\,\mathrm{arctg}\left(\frac{e^x+1110}{3\sqrt{111}10}\right) + c$

5. $\int \frac{dx}{\mathrm{sh}x+2\,\mathrm{ch}x} = \int \frac{e^x dx}{e^{2x}-1+2e^{2x}+2} = \int \frac{e^x dx}{3e^{2x}+1} = \int \frac{da}{3a^2+1} = \int \frac{da}{(\sqrt{3}a)^2+1} =$

$e^x = a$
$e^x dx = da$
$\sqrt{3}\,a = b$
$\sqrt{3}\,da = db$
$da = db/\sqrt{3}$

$= \frac{1}{\sqrt{3}}\int \frac{db}{b^2+1} = \frac{1}{\sqrt{3}}\,\mathrm{arctg}(\sqrt{3}\,e^x) + c$

6. $\int \frac{dx}{2\operatorname{sh}x - \operatorname{ch}x} = \int \frac{e^x dx}{2e^{2x} - 2 - e^{2x} - 1} = \int \frac{e^x dx}{e^{2x} - 3} = \int \frac{da}{a^2 - 3} = \frac{1}{3}\int \frac{da}{\left(\frac{a}{\sqrt{3}}\right)^2 - 1}$

$e^x = a$ $\quad a\sqrt{3} = b$
$e^x dx = da$ $\quad da\sqrt{3} = db$
$da = \sqrt{3}db$

$= \sqrt{3}/3 \int \frac{db}{b^2 - 1} = \frac{\sqrt{3}}{3} \ln\left|\frac{e^x\sqrt{3} - 1}{e^x\sqrt{3} + 1}\right| + C$

7. $\int \frac{dx}{4 + 5\operatorname{ch}x + 3\operatorname{sh}x} = \int \frac{e^x dx}{4 + 5e^{2x} + 5 + 3e^{2x} - 3} = \int \frac{e^x dx}{8e^{2x} + 4e^x + 2} = \frac{1}{2}\int \frac{da}{4a^2 + 2a + 1} =$

$e^x = a$
$e^x dx = da$
$2a = b$
$2da = db$
$da = db/2$

$= \frac{1}{2}\int \frac{da}{(2a+1)^2 - 2a} = \frac{1}{4}\int \frac{db}{(b+1)^2 - b}$

ОВАЛ ПОЛІСВІТИ!

8. $\int \frac{dx}{2 + \operatorname{ch}x + \operatorname{sh}x} = \int \frac{e^x dx}{2e^x + e^{2x} + 1 + e^{2x} - 1} = \frac{1}{2}\int \frac{e^x dx}{e^{2x} + e^x} = \frac{1}{2}\int \frac{da}{a^2 + a} = \frac{1}{2}\int \frac{da}{a(a+1)}$

$e^x = a$
$e^x dx = da$

$1 = A(a+1) + Ba$
$A + B = 0$
$A = 1 \quad B = -1$

$= \frac{1}{2}\left[\int \frac{da}{a} - \int \frac{da}{a+1}\right] =$

$= \frac{x}{2} - \frac{1}{2}\ln(e^x + 1) + C$

9. $\int \frac{dx}{3 + 3\operatorname{ch}x + 5\operatorname{sh}x} = \int \frac{e^x dx}{3e^x + 3e^{2x} + 3 + 5e^{2x} - 5} = \int \frac{e^x dx}{8e^{2x} + 3e^x - 2} = \int \frac{da}{8a^2 + 3a - 2}$

$e^x = a$
$e^x dx = da$

10. $\int \frac{dx}{2\operatorname{sh}x + 3\operatorname{ch}x} = \int \frac{e^x dx}{2e^{2x} - 2 + 3e^{2x} + 3} = \int \frac{e^x dx}{5e^{2x} + 1} = \int \frac{da}{5a^2 + 1} = \int \frac{da}{(\sqrt{5}a)^2 + 1} =$

$e^x = a$ $\quad \sqrt{5}a = b$
$e^x dx = da$ $\quad \sqrt{5}da = db$
$da = db/\sqrt{5}$

$= \frac{\sqrt{5}}{5}\int \frac{db}{b^2 + 1} = \frac{\sqrt{5}}{5}\operatorname{arctg}(\sqrt{5}e^x) + C$

11. $\int \frac{\operatorname{sh}x + 2\operatorname{ch}x}{2\operatorname{sh}x - \operatorname{ch}x - 1} dx = \int \frac{e^{2x} - 1 + 2e^{2x} + 2}{2e^x} \cdot \frac{2e^x}{2e^{2x} - 2 - e^{2x} - 1 - e^x} dx =$

$= \int \frac{3e^{2x} + 1}{e^{2x} - e^x - 3} dx = \int \frac{e^{2x}\left(3 + \frac{1}{e^{2x}}\right)}{e^{2x}\left(1 - \frac{1}{e^x} - \frac{3}{e^{2x}}\right)} dx$

$\int \frac{\operatorname{ch}^2 x (\operatorname{th}^2 x + 2)\, dx}{\operatorname{ch}^2 x \left(2\operatorname{th}^2 x - 1 - \frac{1}{\operatorname{ch}^2 x}\right)}$

12. $\int \frac{sh^2x}{1-sh^2x}dx = \int \frac{sh^2x\,dx}{2+ch^2x} = \int \left(\frac{e^{2x}-1}{2e^x}\right)^2 \cdot \left(\frac{2e^x}{2e^x - e^{2x}+1}\right)^2 dx =$

$ch^2x - 1$

13. $\int sh^2x\,ch^3x\,dx = \int (sh^2x\,ch^2x\,ch x\,dx = \int sh^2x\,(1+sh^2x)\,ch x\,dx =$

$= \int (sh^2x + sh^4x)\,ch x\,dx = \int (a^2+a^4)\,da = \int a^2 da + \int a^4 da = \frac{sh^3x}{3} + \frac{sh^5x}{5} + C$

$shx = a$
$ch x\,dx = da$

14. $\int ch^3x\,sh^8x\,dx = \int ch^2x\,sh^8x\,ch x\,dx = \int (1+sh^2x)\,sh^8x\,ch x\,dx = \int (1+a^2)a^8\,da =$

$shx = a$
$ch x\,dx = da$
$= \int (a^8 + a^{10})da = \int a^8 da + \int a^{10} da = \frac{sh^9x}{9} + \frac{sh^{11}x}{11} + C$

15. $\int sh^3x\,ch2x\,dx = \int \frac{(e^{2x}-1)^3}{8e^{3x}} \cdot \frac{e^{4x}+1}{2e^{2x}}dx =$

16. $\int \frac{dx}{shx\,chx} = \int \frac{2e^x \cdot e^x \cdot 2}{(e^{2x}-1)(e^{2x}+1)}dx = 4\int \frac{e^{2x}}{e^{4x}-1}dx = 2\int \frac{e^a}{e^{2a}-1}da =$

$2x = a \quad e^a = b$
$2dx = da \quad e^a da = db$
$dx = da/2$
$= 2\int \frac{db}{b^2-1} = 2\,\mathrm{fn}\left|\frac{e^{2x}-1}{e^{2x}+1}\right| + C$

17. $\int \frac{dx}{shx\,ch^2x} = \int \frac{ch^2x - sh^2x}{shx\,ch^2x}dx = \int \frac{dx}{shx} - \int \frac{shx\,dx}{ch^2x} =$

$= \int \frac{2e^x dx}{e^{2x}-1} - \int \frac{da}{a^2} = 2\int \frac{db}{b^2-1} + \frac{1}{chx} = 2\,\mathrm{fn}\left|\frac{e^x-1}{e^x+1}\right| + \frac{1}{chx} + C$

$chx = a \quad e^x = b$
$shx\,dx = da \quad e^x dx = db$

18. $\int \frac{dx}{sh^4x\,ch^2x} = \int \frac{ch^2x - sh^2x}{sh^4x\,ch^2x}dx = \int \frac{dx}{sh^4x} - \int \frac{dx}{sh^2x\,ch^2x} = \int \frac{ch^2x - sh^2x}{sh^4x}dx -$

$- \int \frac{ch^2x - sh^2x}{sh^2x\,ch^2x}dx = \int \frac{ch^2x}{sh^4x}dx - \int \frac{dx}{sh^2x} - \int \frac{dx}{sh^2x} + \int \frac{dx}{ch^2x} =$

$= \int \frac{ch^2x}{sh^2x} \cdot \frac{dx}{sh^2x} + cthx + cthx + thx = \int cth^2x \frac{dx}{sh^2x} + 2cthx + thx =$

$a = cthx$
$da = \frac{-dx}{sh^2x}$
$= -\int a^2 da + 2cthx + thx = -\frac{1}{3}cth^3x + 2cthx + thx + C$

$\frac{dx}{sh^2x} = -da$

19. $\int \frac{dx}{sh^3 x ch^2 x} = \int \frac{ch^2 x - sh^2 x}{sh^3 x ch^2 x} dx = \int \frac{dx}{sh^3 x} - \int \frac{dx}{sh x ch^2 x} =$

$= \int \frac{ch^2 x - sh^2 x}{sh^3 x} dx - \int \frac{ch^2 x - sh^2 x}{sh x ch^2 x} dx = \underbrace{\int \frac{ch^2 x}{sh^3 x} dx - \int \frac{dx}{sh x}} - \int \frac{dx}{sh x} + \int \frac{sh x dx}{ch^2 x} =$

$= -2\int \frac{2e^x dx}{e^{2x}-1} + \int \frac{da}{a^2} = -4\int \frac{db}{b^2-1} - \frac{1}{ch x} =$

$ch x = a \quad e^x = b$
$sh x dx = da \quad e^x dx = db$

$= -4 \ln \left| \frac{e^x - 1}{e^x + 1} \right| - \frac{1}{ch x} + C$

20. $\int \frac{dx}{(1+ch x)^2} = \int \left(\frac{2e^x}{2e^x + e^{2x} + 1} \right)^2 dx = \int \frac{4a}{(2a + a^2 + 1)^2} da =$

$e^x = a$
$e^x dx = da$

$= \int \frac{4a}{(a+1)^2} da = 4 \ln (e^x + 1) + \frac{1}{e^x + 1} + C$

$4a = Aa + A + B$
$A = 4 \quad A + B = 0$
$B = -4$

21. $\int \frac{sh^4 x}{ch^3 x} dx = \int \frac{(ch^2 x - 1)^2}{ch^3 x} dx = \int \frac{ch^4 x - 2ch^2 x + 1}{ch^3 x} dx =$

$= \int ch x dx - 2\int \frac{dx}{ch x} + \int \frac{dx}{ch^3 x} = sh x - 2\int \frac{2e^x dx}{e^{2x}+1} + \int \frac{ch^2 x - sh^2 x}{ch^3 x} dx =$

$= sh x - 4\int \frac{da}{a^2+1} + \int \frac{dx}{ch x} - \int \frac{sh^2 x}{ch^3 x} dx =$

$e^x = a$
$e^x dx = da$

$= sh x - 4 \operatorname{arctg} e^x$

22. $\int \frac{sh^4 x}{ch^6 x} dx = \int \left(\frac{sh x}{ch x} \right)^4 \frac{dx}{ch^2 x} = \int th^4 x \frac{dx}{ch^2 x} = \int a^4 da = \frac{th^5 x}{5} + C$

$th x = a$
$\frac{dx}{ch^2 x} = da$

23. $\int \frac{ch^5 x}{sh x} dx = \int \frac{(1+sh^2 x)^2 ch x}{sh x} dx = \int \frac{(1+a^2)^2}{a} da =$

$sh x = a$
$ch x dx = da$

$= \int \frac{1 + 2a^2 + a^4}{a} da = \int \frac{da}{a} + 2\int a da + \int a^3 da =$

$= \ln |sh x| + sh^2 x + \frac{sh^4 x}{4} + C$

$\int \frac{ch^2x}{sh^3x}dx = \int \frac{1+sh^2x}{sh^3x}dx = \int$

$\left(\frac{e^{2x}+1}{2e^x}\right)^2 \cdot \left(\frac{2e^x}{e^{2x}-1}\right)^3 dx = \int \frac{(e^{2x}+1)^2}{(e^{2x}-1)^3} \cdot 2e^x dx = 2\int \frac{(a^2+1)^2}{(a^2-1)^3}da =$

$e^x = a$
$e^x dx = da$
$= 2\int \frac{a^4+2a^2+1}{(a+1)^3(a-1)^3}da$

25. $\int ch^3x \sqrt[3]{sh^2x}\,dx = \int (1+sh^2x)\sqrt[3]{sh^2x}\,ch x dx = \int (1+a^2)\cdot a^{2/3}da =$
$sh x = a$
$ch x dx = da$
$= \int a^{2/3}da + \int a^{8/3}da = \frac{3}{5}(sh x)^{5/3} + \frac{3}{11}(sh x)^{11/3} + C$

26. $\int \frac{sh^3x}{\sqrt[3]{ch^2x}}dx = \int \frac{(ch^2x-1)\,sh x}{\sqrt[3]{ch^2x}}dx = \int \frac{a^2-1}{a^{2/3}}da = \int a^{4/3}da -$
$ch x = a$
$sh x dx = da$
$- \int a^{-2/3}da = \frac{3}{7}(ch x)^{7/3} - 3\sqrt[3]{ch x} + C$

27. $\int \frac{\sqrt[3]{tg^2x}}{ch^4x}dx$

tg

28. $\int th^3x dx = \int \frac{sh^3x}{ch^3x}dx = \int \frac{ch^2x-1}{ch^3x}sh x dx = \int \frac{a^2-1}{a^3}da = \int \frac{da}{a} - \int a^{-3}da =$
$ch x = a$
$sh x dx = da$
$= \ln|ch x| + \frac{1}{2ch^2x} + C$

29. $\int th^4x dx = \int \frac{sh^4x}{ch^4x}dx =$

31. $\int sh x \cos x dx =$
$u = \cos x \quad du = -sm x dx$
$dv = \int e$
$u = sh x \quad du = ch x dx \quad sh x \cdot sm x - \int sm x\, ch x dx =$
$dv = \cos x dx \quad v = sm x$
$u = ch x \quad du = sh x dx \quad = sh x \cdot sm x + ch x \cdot \cos x - \underbrace{\int sh x \cos x dx}_{J}$
$dv = sm x \quad v = -\cos x$
$J = \frac{1}{2}(sh x \sin x + ch x \cos x) + C$

32. $\int ch x \cos x dx = ch x sm x - \int sm x\, sh x dx = ch x sm x + sh x \cos x -$
$u = ch x \quad du = sh x dx$
$dv = \cos x dx \quad v = sm x \quad - \int ch x \cos x dx$
$u = sh x \quad du = ch x dx \quad J = \frac{1}{2}(ch x sm x + sh x \cos x) + C$
$dv = sm x dx \quad v = -\cos x$

133. $\int \sin^2 x \cos^3 x\,dx = \int \sin^2 x(\cos^2 x \cos x - \sin^2 x \sin x)\,dx =$

$= \int \sin^2 x(\cos^3 x - \sin^2 x \cos x - 2\sin^2 x \cos x)\,dx =$

$= \int \sin^2 x(\cos^3 x - 3\sin^2 x \cos x)\,dx = \int \sin^2 x \cos x(1 - 4\sin^2 x)\,dx$

$u = \sin^2 x \quad du = 2\sin x \cos x\,dx$

$dv = \cos x\,dx \Rightarrow v = \frac{1}{3}\sin 3x$

$\frac{1}{3}\sin^2 x \sin 3x - \frac{2}{3}\int \sin x \sin 3x\,dx =$

$= \frac{1}{3}\sin^2 x \sin 3x + \frac{2}{9}\sin x \cos 3x - \frac{4}{9}\int \sin x \cos 3x\,dx =$

$u = \sin x \quad du = \cos x\,dx$

$dv = \sin 3x\,dx \quad v = -\frac{1}{3}\cos 3x$

J

$J = \frac{9}{13}\left[\frac{1}{3}\sin^2 x \sin 3x + \frac{2}{9}\sin x \cos 3x\right] + C$