Зад. 5. Одредити 1. извод функције $y=f(x)$ која је дата имплицитно:

а) $x^3+y^3-8=0 \quad /'$

Решење: $3x^2+3y^2\cdot y'=0 \quad \Rightarrow y'=-\frac{x^2}{y^2}$

б) $e^{\frac{y}{x}}\cdot\cos\left(\frac{y}{x}\right)=1 \quad /'$

Решење: $\left(e^{\frac{y}{x}}\right)'\cdot\cos\left(\frac{y}{x}\right)+\left(e^{\frac{y}{x}}\right)\cdot\left(\cos\left(\frac{y}{x}\right)\right)'=0$

$e^{\frac{y}{x}}\cdot\left(\frac{y}{x}\right)'\cdot\cos\left(\frac{y}{x}\right)+e^{\frac{y}{x}}\cdot\left(-\sin\left(\frac{y}{x}\right)\right)\cdot\left(\frac{y}{x}\right)'=0$

$\left(\frac{y}{x}\right)'\left(\cos\frac{y}{x}-\sin\frac{y}{x}\right)=0$

$\frac{y'x-y}{x^2}=0 \quad \Rightarrow y'x-y=0 \quad \Rightarrow y'=\frac{y}{x}$

Вежба бр. 11 — 15. 03. 2007.

Лопиталово правило

Деф. 1. Нека су ф-је f и g диференцијабилне при чему је $g'(x)\neq 0$, и $\lim\limits_{x\to a_+} f(x)=\lim\limits_{x\to a_+} g(x)=0$ и $\exists \lim\limits_{x\to a_+}\frac{f'(x)}{g'(x)}=A$. Тада $\exists \lim\limits_{x\to a_+}\frac{f(x)}{g(x)}=A$

Ово правило примењујемо при облику $\frac{0}{0}$, $\frac{\infty}{\infty}$

Зад. 1. $\lim\limits_{x\to 0_+}\frac{\ln(1-\cos x)}{\ln \operatorname{tg} x} \Leftrightarrow * \longrightarrow \frac{\infty}{\infty}$

Решење: $* \overset{\text{Л.П.}}{=} \lim\limits_{x\to 0_+}\frac{\frac{1}{1-\cos x}\cdot\sin x}{\frac{1}{\operatorname{tg}x}\cdot\frac{1}{\cos^2 x}}=\lim\limits_{x\to 0+}\frac{\frac{\sin x}{1-\cos x}}{\frac{1}{\sin x\cdot\cos x}}=\lim\limits_{x\to 0+}\frac{\sin^2 x}{2\sin^2\frac{x}{2}}=\lim\limits_{x\to 0+}\frac{\left(\frac{\sin x}{x}\right)^2\cdot x^2}{2\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\cdot\frac{x^2}{4}}=2$

Зад. 2. $\lim\limits_{x\to 0}\frac{\sqrt[3]{\cos 3x}-\sqrt[3]{\cos 5x}}{x^2} \Leftrightarrow * \to \frac{0}{0}$

Решење: $* \overset{\text{Л.П.}}{=} \lim\limits_{x\to 0}\frac{\frac{1}{3}(\cos 3x)^{-\frac{2}{3}}\cdot(-\sin 3x)\cdot 3-\frac{1}{3}(\cos 5x)^{-\frac{2}{3}}\cdot(-\sin 5x)\cdot 5}{2x}$

$=-\frac{3}{3}\lim\limits_{x\to 0}\frac{(\cos 3x)^{-\frac{2}{3}}\cdot\left(\frac{\sin 3x}{3x}\right)\cdot 3x}{2x}+\frac{5}{3}\lim\limits_{x\to 0}\frac{(\cos 5x)^{-\frac{2}{3}}\left(\frac{\sin 5x}{5x}\right)\cdot 5x}{2x}$

$=-\frac{9}{10}+\frac{25}{6}=\frac{49}{15}$

Зад. 3. $\lim\limits_{x\to 0}\frac{\ln(\cos 5x)}{\ln(\cos 9x)} \Leftrightarrow * \to \frac{0}{0}$

Решење: $* \overset{\text{Л.П.}}{=} \lim\limits_{x\to 0}\frac{\frac{1}{\cos 5x}\cdot(-\sin 5x)\cdot 5}{\frac{1}{\cos 9x}\cdot(-\sin 9x)\cdot 9}=\frac{5}{9}\lim\limits_{x\to 0}\frac{\frac{\sin 5x}{5x}\cdot 5x}{\frac{\sin 9x}{9x}\cdot 9x}=\frac{25}{81}$

Зад. 4. $\lim\limits_{x\to+\infty}(\pi-2\operatorname{arctg}x)\cdot x \Leftrightarrow * \to 0\cdot\infty$

Решење: $*=\lim\limits_{x\to+\infty}\frac{(\pi-2\operatorname{arctg}x)}{\frac{1}{x}} \overset{\text{Л.П.}}{=} \lim\limits_{x\to+\infty}\frac{-2\cdot\frac{1}{1+x^2}}{-\frac{1}{x^2}}=\lim\limits_{x\to+\infty}\frac{2x^2}{1+x^2}=2$

Зад. 5. $\lim\limits_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) \Leftrightarrow * \to \infty-\infty$

Решење: $* = \lim\limits_{x\to 1}\frac{x-1-\ln x}{(x-1)\cdot\ln x} \nearrow \frac{0}{0} \overset{\text{Л.П.}}{=} \lim\limits_{x\to 1}\frac{1-\frac{1}{x}}{\ln x+(x-1)\frac{1}{x}} \nearrow \frac{0}{0} = \lim\limits_{x\to 1}\frac{\frac{x-1}{x}}{\frac{x\ln x+x-1}{x}} =$

$$= \lim_{x\to 1}\frac{x-1}{x\ln x+x-1} \nearrow \frac{0}{0} \overset{\text{Л.П.}}{=} \lim_{x\to 1}\frac{1}{\ln x+1+1} = \frac{1}{2}$$

Зад. 6. $\lim\limits_{x\to\frac{\pi}{2}}(\sin x)^{\operatorname{tg}x} = L \;/\ \ln$

Решење: $\ln L = \lim\limits_{x\to\frac{\pi}{2}}\ln(\sin x)^{\operatorname{tg}x} = \lim\limits_{x\to\frac{\pi}{2}}\operatorname{tg}x\cdot\ln(\sin x) \nearrow \infty\cdot 0 = \lim\limits_{x\to\frac{\pi}{2}}\frac{\ln(\sin x)}{\operatorname{ctg}x} \nearrow \frac{0}{0}$

$$\overset{\text{Л.П.}}{=} \lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{\sin x}\cdot\cos x}{-\frac{1}{\sin^2 x}} = 0, \quad \ln L = 0, \quad L = e^0 = 1$$

Зад. 7. $\lim\limits_{x\to 0}\left(\frac{\operatorname{tg}x}{x}\right)^{\frac{1}{x^2}} = L \;/\ \ln$

Решење: $\ln L = \lim\limits_{x\to 0}\ln\left(\frac{\operatorname{tg}x}{x}\right)^{\frac{1}{x^2}} = \lim\limits_{x\to 0}\frac{\ln\left(\frac{\operatorname{tg}x}{x}\right)}{x^2} \nearrow \frac{0}{0}$

$$\overset{\text{Л.П.}}{=} \lim_{x\to 0}\frac{\frac{x}{\operatorname{tg}x}\cdot\frac{\frac{1}{\cos^2 x}\cdot x-\operatorname{tg}x}{x^2}}{2x} = \lim_{x\to 0}\frac{\frac{x-\sin x\cdot\cos x}{\cos^2 x \to 1}}{2x^2\operatorname{tg}x = \frac{\sin x}{\cos x \to 1}}$$

$$= \lim_{x\to 0}\frac{x-\sin x\cdot\cos x}{2x^2\cdot\sin x} \nearrow \frac{0}{0} \overset{\text{Л.П.}}{=} \lim_{x\to 0}\frac{1-\cos^2 x+\sin^2 x}{2(2x\sin x+x^2\cos x)}$$

$$= \lim_{x\to 0}\frac{2\sin^2 x \quad /:x^2}{2(2x\sin x+x^2\cos x)/:x^2} = \lim_{x\to 0}\frac{\left(\frac{\sin x}{x}\right)^2}{2\frac{\sin x}{x}+\cos x} = \frac{1}{3} \Rightarrow L = e^{\frac{1}{3}}$$

Изводи вишег реда

Теор. 1. (Лајбницова формула) Ако су u и v n пута диференцијабилне функције, онда је и $u\cdot v$ n пута диференцијабилна ф-ја и важи

$$(u\cdot v)^{(n)} = \sum_{k=0}^{n}\binom{n}{k}u^{(n-k)}\cdot v^{(k)} = \binom{n}{0}u^{(n)}\cdot v+\binom{n}{1}u^{(n-1)}\cdot v^{(1)}+\dots+\binom{n}{n}u\cdot v^{(n)}$$

Зад. 1. $f(x) = x^2\cdot e^x$, $f^{(n)}(x) = ?$

Решење: $f^{(n)}(x) = \sum\limits_{k=0}^{n}\binom{n}{k}u^{(n-k)}\cdot v^{(k)}$ $\quad u = x^2$, $v = e^x$

$$f^{(n)}(x) = \binom{n}{0}(e^x)^{(n)}\cdot x^2+\binom{n}{1}(e^x)^{(n-1)}(x^2)'+\binom{n}{2}(e^x)^{(n-2)}(x^2)''+\binom{n}{3}\cancel{(e^x)^{(n-3)}}(x^2)''' \to 0 \dots$$

$$= e^x\cdot x^2+n\cdot e^x\cdot 2x+\frac{n(n-1)}{\cancel{2}}\cdot e^x\cdot\cancel{2}$$

Зад. 2. $f(x) = (x^3-2x+2)\sin x$, $f^{V}(x) = ?$

Решење: $f^{V}(x) = \sum\limits_{k=0}^{5}\binom{5}{k}(\sin x)^{(5-k)}\cdot(x^3-2x+2)^{(k)}$

$$= \binom{5}{0}(\sin x)^{(5)}(x^3-2x+2)+\binom{5}{1}(\sin x)^{(4)}(x^3-2x+2)'+\binom{5}{2}(\sin x)^{(3)}(x^3-2x+2)''$$

$$+\binom{5}{3}(\sin x)^{(2)}\cdot(x^3-2x+2)'''+0+0$$

$$= \cos x\,(x^3-2x+2)+5\cdot\sin x\cdot(3x^2-2)+\frac{5\cdot 4}{2\cdot 1}(-\cos x)\cdot 6x+\frac{5\cdot 4\cdot 3}{3\cdot 2\cdot 1}(-\sin x)\cdot 6$$

$$= \cos x\,(x^3-62x+2)+\sin x\,(15x^2-70)$$

$\sin x$, $(\sin x)' = \cos x$, $(\sin x)'' = -\sin x$, $(\sin x)''' = -\cos x$, $(\sin x)^{IV} = \sin x$, $(\sin x)^{V} = \cos x$

Зад. 3. а) $f(x) = x^{\alpha}$

Решење: $f'(x) = \alpha \cdot x^{\alpha-1}$

$f''(x) = \alpha(\alpha-1)x^{\alpha-2}$

$\vdots$

$f^{(n)} = \alpha(\alpha-1)\cdots(\alpha-n+1)x^{\alpha-n}$

б) $f(x) = a^x$

Решење: $f'(x) = a^x \cdot \ln a$

$f''(x) = a^x \cdot \ln^2 a$

$\vdots$

$f^{(n)}(x) = a^x \cdot (\ln a)^n$

в) $f(x) = \sin x$

Решење: $f'(x) = \cos x = \sin\left(x + \frac{\pi}{2}\right)$

$f''(x) = -\sin x = \sin\left(x + \frac{2\pi}{2}\right)$

$\vdots$

$f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right)$

г) $f(x) = \cos x$

Решење: $f'(x) = -\sin x = \cos\left(x + \frac{\pi}{2}\right)$

$f''(x) = -\cos x = \cos\left(x + \frac{2\pi}{2}\right)$

$\vdots$

$f^{(n)}(x) = \cos\left(x + \frac{n\pi}{2}\right)$

д) $f(x) = \ln(1+x)$

Решење: $f'(x) = \frac{1}{1+x} = (1+x)^{-1}$

$f''(x) = -1 \cdot (1+x)^{-2}$

$f'''(x) = 1 \cdot 2 \cdot (1+x)^{-3}$

$f^{IV}(x) = -1 \cdot 2 \cdot 3 \cdot (1+x)^{-4}$

$\vdots$

$f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{(1+x)^n}$

ђ) $f(x) = \frac{1}{x}$

Решење: $f'(x) = -\frac{1}{x^2}$

$f''(x) = 2 \cdot \frac{1}{x^3}$

$f'''(x) = -1 \cdot 2 \cdot 3 \cdot \frac{1}{x^4}$

$f^{IV}(x) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \frac{1}{x^5}$

$\vdots$

$f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}}$

Зад. 4. $f(x) = \frac{3x-1}{x^2-3x+2} = \frac{3x-1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$

Решење: $3x-1 = A(x-2) + B(x-1) = x(A+B) + (-2A-B)$ $\quad A=-2 \quad B=5 \quad f^{(n)}(x) = ?$

$f^{(n)}(x) = -2 \cdot (-1)^n \frac{n!}{(x-1)^{n+1}} + 5(-1)^n \frac{n!}{(x-2)^{n+1}}$

вежба бр. 12 16.03.2007.

Испитивање функције и њен график

1° Област дефинисаности (домен) $\mathcal{D}_f$

2° Парност функције

$f(-x) = f(x) \quad \forall x \in \mathcal{D}_f$ парна ф-ја (симетрична на y-осу)
$f(-x) = -f(x) \quad \forall x \in \mathcal{D}_f$ непарна ф-ја (симетрична на координатни почетак)

3° Нуле и знак ф-је

$f(x) = 0 \qquad f(x) \gtrless 0$

4° Асимптоте и понашање ф-је на крајевима интервала дефинисаности

а) В.А. $\mathcal{D}_f = (-\infty, a) \cup (a, +\infty)$ испитујемо у тачкама прекида $\mathcal{D}_f$
$\lim\limits_{x \to a-0} f(x) = -\infty$ ✓ са те стране ф-ја има ВА $\quad \lim\limits_{x \to a+0} f(x) = A$ ✗

б) Х.А. $\lim\limits_{x \to +\infty} f(x) = A$ ✓ $\Rightarrow$ права $y = A$ је ХА $\quad \lim\limits_{x \to -\infty} f(x) = +\infty$ ✗

в) К.А. проверавамо када ф-ја нема Х.А.
$y = kx + n \quad k = \lim\limits_{x \to \pm\infty} \frac{f(x)}{x} \quad n = \lim\limits_{x \to \pm\infty} (f(x) - kx)$

5° Екстремне вредности и интервали монотоности

$f'(x) = 0$ стационарне тачке (могуће екстремне
$f'(x) > 0$ ф-ја монотоно растућа ↗
$f'(x) < 0$ ф-ја монотоно опадајућа ↘

6° Превојне тачке, конвексност и конкавност

$f''(x) = 0$ превојне тачке
$f''(x) > 0$ конвексна $\smile$
$f''(x) < 0$ конкавна $\frown$

7° График ф-је

Зад. 1. Скицирати график ф-је $f(x) = \frac{x^3}{2(x+1)^2}$

Решење: 1° област дефинисаности
$x+1 \neq 0, \quad x \neq -1 \qquad \mathcal{D}_f = (-\infty, -1) \cup (-1, +\infty)$

2° парност функције
ако домен није симетричан онда ф-ја не може имати ни својство парности ни својство непарности

3° Нула и знак ф-је

$f(x)=0 \iff x^3=0,\ x=0$

$A(0,0)$

		-1		0	
x^3	$-$		$-$	0	$+$
$(x+1)^2$	$+$	0	$+$		$+$
$f(x)$	$-$	н.д.	$-$	0	$+$

4° Асимптоте

а) В.А. $\lim\limits_{x\to -1-0} f(x)=\dfrac{(-1)^3}{2(-1-0+1)^2}=\dfrac{-1}{0_+}=-\infty$

$\lim\limits_{x\to -1+0} f(x)=\dfrac{(-1)^3}{2(-1+0+1)^2}=\dfrac{-1}{0_+}=-\infty$

права $x=-1$ је ВА

б) Х.А. $\lim\limits_{x\to\pm\infty}\dfrac{x^3}{2(x+1)^2}=\pm\infty$

нема ХА

в) К.А. $y=kx+m$

$k=\lim\limits_{x\to\pm\infty}\dfrac{f(x)}{x}=\lim\limits_{x\to\pm\infty}\dfrac{x^2}{2(x+1)^2}=\dfrac{1}{2}$

$m=\lim\limits_{x\to\pm\infty}(f(x)-kx)=\lim\limits_{x\to\pm\infty}\left(\dfrac{x^3}{2(x+1)^2}-\dfrac{x}{2}\right)=\lim\limits_{x\to\pm\infty}\dfrac{x^3-x^3-2x^2-x}{2(x+1)^2}=-1$

права $y=\frac{1}{2}x-1$ је К.А.

5° Екстремне вредности и монотоност

$f'(x)=\dfrac{1}{2}\,\dfrac{3x^2(x+1)^2-x^3\cdot 2(x+1)}{(x+1)^4}=\dots=\dfrac{x^2(x+3)}{2(x+1)^3}$

$f'(x)=0 \iff x^2(x+3)=0$

$x=0 \lor x=-3 \qquad M\left(-3,-\frac{27}{8}\right)_{MAX}$

		-3		-1		0	
$x+3$	$-$	0	$+$		$+$		$+$
x^2	$-$		$-$	0	$+$		$+$
$f'(x)$	$+$	0	$-$	НД	$+$	0	$+$
$f(x)$	↗	$-\frac{27}{8}$	↘	НД	↗		↗

6° Превојне тачке

$f''(x)=\dfrac{1}{2}\,\dfrac{(x^3+3x)'\cdot(x+1)^3-(x^3+3x)((x+1)^3)'}{(x+1)^6}=\dots=\dfrac{3x}{(x+1)^4}$ $(x+1)^4>0$

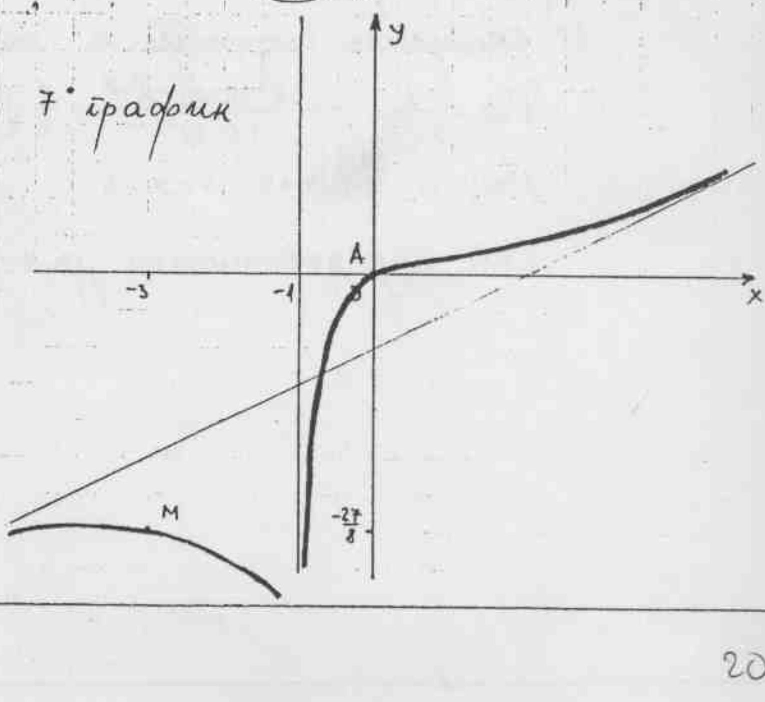
$f''(x)=0 \iff x=0$

		-1		0	
x	$-$		$-$	0	$+$
$f''(x)$	$-$	Н.Д.	$-$	0	$+$
$f(x)$	$\frown$	Н.Д.	$\frown$	0	$\smile$

$f_P=f(0)=0$

$A(0,0)$

7° график



Зад. 2. $f(x)=\sqrt{\frac{x^3}{x-2}} = |x|\sqrt{\frac{x}{x-2}}$

Решење: 1° Област дефинисаности

$\frac{x^3}{x^2-2} \geq 0$

	0		2	
x^3	$-$	$+$		$+$
$x-2$	$-$	$-$		$+$
	$+$	$-$		$+$

$\mathcal{D}_f = (-\infty, 0] \cup (2, +\infty)$

2° Парност функције

ф-ја није ни парна ни непарна јер домен није симетричан

3° Нуле и знак ф-је

$f(x)=0 \iff x=0$, $A(0,0)$ $\qquad f(x)>0$, $\forall x \in \mathcal{D}_f$

4° Асимптоте

а) В.А. $\lim_{x\to 2+0} \sqrt{\frac{x^3}{x-2}} = \sqrt{\frac{8}{2+0-2}} = +\infty$

права $x=2$ је В.А. са десне стране

б) Х.А. $\lim_{x\to\pm\infty} \sqrt{\frac{x^3}{x-2}} = +\infty$ нема ХА

в) К.А. $y=kx+n$

$k=\lim_{x\to\pm\infty}\frac{f(x)}{x} = \lim_{x\to\pm\infty}\frac{|x|\overset{1}{\sqrt{\frac{x}{x-2}}}}{x} = \lim_{x\to\pm\infty} sgn\, x = \pm 1$

$n_1 = \lim_{x\to+\infty}(f(x)-k_1x) = \lim_{x\to+\infty}\left(\sqrt{\frac{x^3}{x-2}}-x\right)\cdot\frac{\sqrt{\frac{x^3}{x-2}}+x}{\sqrt{\frac{x^3}{x-2}}+x} = \lim_{x\to+\infty}\frac{2x^2}{2x(x-2)} = 1$

$n_2 = \lim_{x\to-\infty}\left(|x|\sqrt{\frac{x}{x-2}}+x\right) = \lim_{x\to-\infty}\left(-x\sqrt{\frac{x}{x-2}}+x\right) = \dots = -1$

праве $y=x+1$ $(x\to+\infty)$ и $y=-x-1$ $(x\to-\infty)$ су К.А.

5° Екстремне вредности и монотоност

$f'(x) = \frac{1}{2\sqrt{\frac{x^3}{x-2}}}\cdot\frac{3x^2(x-2)-x^3\cdot 1}{(x-2)^2} = \frac{1}{2}\sqrt{\frac{x-1}{x^3}}\cdot\frac{x^2(x-3)}{(x-2)^2} > 0$

$f'(x)=0 \iff x=0 \lor x=3$

$f'(x)$ није дефинисана за $x=2$

		0		2		3	
$x-3$	$-$				$-$	0	$+$
$f'(x)$	$-$	0		Н.Д.	$-$	0	$+$
$f(x)$	$\searrow$	0		Н.Д.	$\searrow$	0	$\nearrow$

$M(3, 3\sqrt{3})_{MIN}$

6° Конвексност, конкавност и превојне тачке

$$f''(x) = \frac{1}{2}\left(\sqrt{\frac{x-2}{x^3}}\cdot\frac{x^3-3x^2}{(x-2)^2}\right)'$$

$$= \frac{1}{2}\left(\frac{1}{2}\sqrt{\frac{x^3}{x-2}}\left(\frac{x-2}{x^3}\right)'\cdot\frac{x^3-3x^2}{(x-2)^2} + \sqrt{\frac{x-2}{x^3}}\,\frac{(3x^2-6x)(x-2)^2-x^2(x-3)(2(x-2))}{(x-2)^4}\right)$$

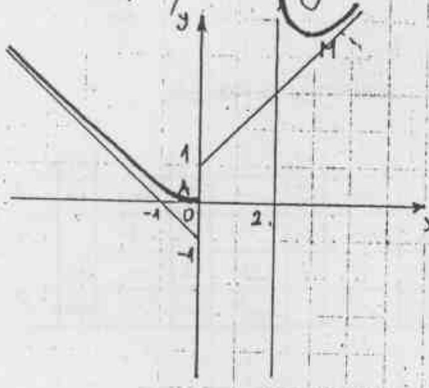
$$= \frac{1}{2}\sqrt{\frac{x^3}{x-2}}\left(\frac{1}{2}\cdot\frac{x^3-(x-2)\cdot 3x^2}{x^6}\cdot\frac{x^2(x-3)}{(x-2)^2} + \frac{x-2}{x^3}\cdot\frac{x(x-2)((3x-6(x-2)-2x(x-3))}{(x-2)^4}\right)$$

$$= \frac{1}{2}\sqrt{\frac{x^3}{x-2}}\left(\frac{1}{2}\cdot\frac{x^2(x-3x+6)(x-3)}{x^4(x-2)^2} + \frac{3x^2-6x-6x+12-2x^2+6x}{x^2(x-2)^2}\right)$$

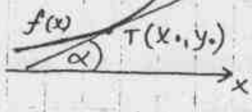
$$= \frac{1}{2}\sqrt{\frac{x^3}{x-2}}\left(\frac{-x^2+6x-9+x^2-6x+12}{x^2(x-2)^2}\right)$$

$$= \frac{1}{2}\sqrt{\frac{x^3}{x-2}}\;\frac{3}{x^2(x-2)^2} > 0 \Rightarrow$$ ф-ја увек конвексна

7° Графикон ф-је



8° Угао под којим ф-ја улази у 0



$$tg\alpha = f'(x_0) = \lim_{x\to x_0} f'(x)$$

$x_0 = 0$

како се ф-ја приближава тачки прекида

$$f'(x) = \frac{1}{2}\,\frac{(x-2)^{\frac{1}{2}}}{x^{3/2}}\cdot\frac{x^2(x-3)}{(x-2)^2}$$

$$= \frac{1}{2}\,\frac{x^{\frac{1}{2}}(x-3)}{(x-2)^{3/2}} = \frac{1}{2}\sqrt{\frac{x}{x-2}}\cdot\frac{x-3}{x-2}$$

$f'(0) = 0 \Rightarrow tg\alpha = 0$

$\alpha = 0 \Rightarrow O_x$ оса је лева полутангента ф-је $f(x)$ у тачки 0

вежба бр. 13 20.03.2007.

Наставак

Зад. 3. $f(x) = \sqrt[3]{x^3+3x^2}$

Решење: 1° Област дефинисаности

$D_f = R$

2° Парност функције

$$f(-x) = \sqrt[3]{(-x)^3+3(-x)^2} = \sqrt[3]{-x^3+3x^2} = -\sqrt[3]{x^3-3x^2}$$

3° Нуле ф-је и знак

$f(x)=0 \Leftrightarrow x^3+3x^2=0$

$x^2(x+3)=0$

$x=0 \vee x=-3$

$A(0,0)$ $B(-3,0)$

-3 $\quad$ 0

$x+3$	−	+	+
x^2	+	+	+
$f(x)$	−	+	+

4° Асимптоте

а) В.А. нема

б) Х.А. $\lim\limits_{x\to\pm\infty} \sqrt[3]{x^3+3x^2} = \pm\infty$ нема ХА

в) К.А. $y = kx+n$

$$k = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{\sqrt[3]{x^3+3x^2}}{x} = 1$$

$$n = \lim_{x\to\pm\infty} (f(x)-kx) = \lim_{x\to\pm\infty} \left(\sqrt[3]{x^3+3x^2} - x\right) \cdot \frac{\sqrt[3]{(x^3+3x^2)^2} - \dots}{\dots}$$

$$= \lim_{x\to\pm\infty} \frac{3x^2}{x^2\left(\sqrt[3]{(1+\frac{3}{x})^2} + \sqrt[3]{1+\frac{3}{x}} + 1\right)} = 1$$

права $y = x+1$ је КА

5° Екстремне вредности и интервали монотоности

$$f'(x) = \dots = \frac{x(x+2)}{(x^3+3x^2)^{\frac{2}{3}}}, \quad (x^3+3x^2)^{\frac{2}{3}} > 0$$

$f'(x) = 0 \iff x = 0 \lor x = -2$

$f_{MAX} = f(-2) = \sqrt[3]{4}$ $\quad f_{MIN} = f(0) = 0$

$M_1(-2, \sqrt[3]{4})$ $\quad M_2(0,0)$

		-3		-2		0	
x	—		—		—	0	+
$x+2$	—		—	0	+		+
$f'(x)$	+	НД	+	0	—	НД	+
$f(x)$	↗		↗	MAX	↘	MIN	↗

6° Превојне тачке и интервали $\cup$ и $\cap$

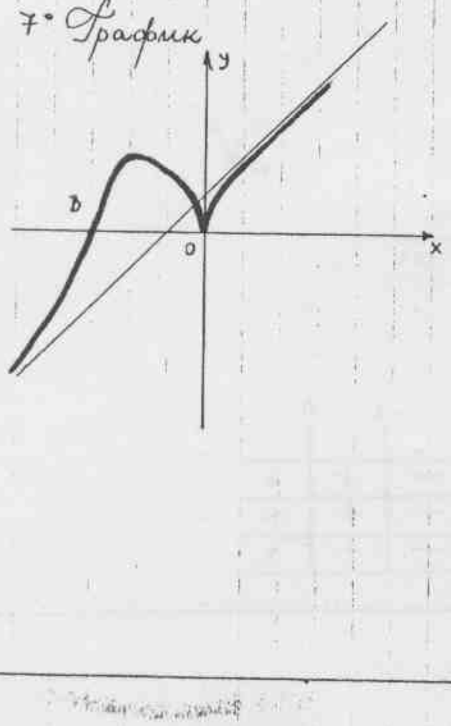
$$f''(x) = \frac{(2x+2)(x^3+3x^2)^{\frac{2}{3}} - (x(x-2)) \cdot \frac{2}{3}(x^3+3x^2)^{-\frac{1}{3}}(3x^2+6x)}{(x^3+3x^2)^{\frac{4}{3}}} = \dots = \frac{-2x^2}{(x^3+3x^2)^{\frac{5}{3}}}$$

$f''(x) = 0 \iff x = 0$

$P(-3, 0)$

		-3		0	
$-2x^2$	—		—		—
$(x^2(x+3))^{\frac{5}{3}}$	—	0	+	0	+
$f''(x)$	+	НД	—	НД	—
$f(x)$	$\cup$	0	$\cap$		$\cup$

7° График



$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{x(x+2)}{x^{4/3}(x+3)^{2/3}} = \infty$$

$\operatorname{tg}\alpha = \infty \implies \alpha = \frac{\pi}{2}$

у-оса је тангента ф-је $f(x)$ у тачки $(0,0)$

Зад. 4. $f(x) = (x+3)\, e^{\frac{1}{x+1}}$

Решење: 1° Област дефинисаности

$x \neq -1 \quad D_f = (-\infty, -1) \cup (-1, +\infty)$

2° Парност ф-је

ф-ја није ни парна ни непарна јер D_f није симетрична

3° Нуле и знак ф-је

$f(x) = 0 \iff x+3=0, \; x=-3 \qquad A(-3,0)$

		-3	-1
$x+3$	−	+	+
$f(x)$	−	+	+

4° Асимптоте

а) В.А. $\lim\limits_{x\to -1-0} (x+3)e^{\frac{1}{x+1}} = (-1+3)e^{\frac{1}{-1-0+1}} = 2\cdot e^{-\infty} = 2\cdot 0_+ = 0$

$\lim\limits_{x\to -1+0} (x+3)e^{\frac{1}{x+1}} = (-1+3)e^{\frac{1}{-1+0+1}} = 2\cdot e^{+\infty} = 2\cdot(+\infty) = +\infty$

$x=1$ је ВА са десне стране

б) Х.А. $\lim\limits_{x\to\pm\infty} (x+3)e^{\frac{1}{x+1}} = \pm\infty\cdot e^{0} = \pm\infty\cdot 1 = \pm\infty$ ↯

в) К.А. $y = kx+m$

$k = \lim\limits_{x\to\pm\infty} \frac{f(x)}{x} = \lim\limits_{x\to\pm\infty} \frac{x+3}{x}\cdot e^{\frac{1}{x+1}} = 1\cdot e^{0} = 1$

$m = \lim\limits_{x\to\pm\infty} (f(x)-kx) = \lim\limits_{x\to\pm\infty} ((x+3)e^{\frac{1}{x+1}} - x) \overset{\infty-\infty}{=} \lim\limits_{x\to\pm\infty} (x(e^{\frac{1}{x+1}}-1) + 3e^{\frac{1}{x+1}})$

$= 3 + \lim\limits_{x\to\pm\infty} x\cdot \frac{e^{\frac{1}{x+1}}-1}{\frac{1}{x+1}}\cdot\frac{1}{x+1} = 4$

права $y = x+4$ КА

5° Екстремне вредности и интервали монотоности

$f'(x) = e^{\frac{1}{x+1}} + (x+3)\cdot e^{\frac{1}{x+1}}\cdot\left(-\frac{1}{(x+1)^2}\right) = \dots = e^{\frac{1}{x+1}}\cdot\frac{x^2+x-2}{(x+1)^2}$

$f'(x) = 0 \iff x^2+x-2=0$

$x=-2 \lor x=1$

		-2		-1		1	
x^2+x-2	+	0	−		−	0	+
$f'(x)$	+	0	−	Н.Д.	−	0	+
	↗	$\frac{1}{e}$ MAX	↘	Н.Д.	↘	$4\sqrt{e}$ MIN	↗

$f_{MAX} = f(-2) = \frac{1}{e}$

$f_{MIN} = f(1) = 4\sqrt{e}$

$M_1\left(-2, \frac{1}{e}\right)$

$M_2\left(1, 4\sqrt{e}\right)$

6° Превојне тачке и интервали $\cup$ и $\cap$

$f''(x) = e^{\frac{1}{x+1}}\left(-\frac{1}{(x+1)^2}\right)\frac{x^2+x-2}{(x+1)^2} + e^{\frac{1}{x+1}}\cdot\frac{(2x+1)(x+1)^2 - (x^2+x-2)\cdot 2(x+1)}{(x+1)^4} = \dots = e^{\frac{1}{x+1}}\frac{5x+7}{(x+1)^4}$

$f''(x) = 0 \iff 5x+7=0, \; x = -\frac{7}{5}$

$f_P = f\left(-\frac{7}{5}\right) = \frac{8}{5}e^{-\frac{5}{2}}$

$P\left(-\frac{7}{5}, \frac{8}{5}e^{-\frac{5}{2}}\right)$

		$-\frac{7}{5}$		-1	
$5x+7$	−	0	+		+
$f''(x)$	−	0	+	Н.Д.	+
$f(x)$	$\cap$	П.Т.	$\cup$	Н.Д.	$\cup$

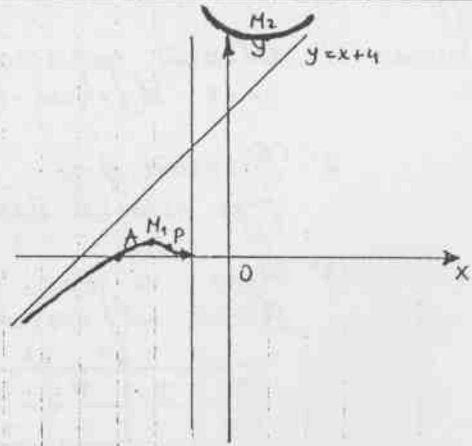
7° График ф-је

$$\lim_{x\to -1-0} f'(x) = \lim_{x\to -1-0} e^{\frac{1}{x+1}} \cdot \frac{x^2+x-2}{(x+1)^2}$$

$$= -2\lim_{x\to -1-0} \frac{e^{\frac{1}{x+1}}}{(x+1)^2} \quad \left[\text{смена: } \frac{1}{1+x} = -t \atop x\to -1-0,\ t\to +\infty\right]$$

$$= -2\lim_{t\to +\infty} \frac{e^{-t}}{\frac{1}{t^2}} = -2\lim_{t\to +\infty} \frac{t^2}{e^t}$$

$$\overset{\text{Л.П.}}{=} -2\lim_{t\to +\infty} \frac{2t}{e^t} \overset{\text{Л.П.}}{=} -4\lim_{t\to +\infty} \frac{1}{e^t} = 0$$



$tg\alpha = 0 \Rightarrow \alpha = 0$

ф-ја се приближава тачки $(-1,0)$ уз x-осу као тангенту

Зад. 5. $f(x) = \ln \frac{x^3}{x^2-1}$

Решење: 1° Област дефинисаности

$\frac{x^3}{x^2-1} > 0 \qquad \mathcal{D}_f = (-1,0) \cup (1,+\infty)$

		-1	0	1	
x^3	−	−	+	+	
x^2-1	+	−	−	+	
	−	+	−	+	

2° Парност ф-је ↯

3° Нуле и знак ф-је

$f(x) = 0 \Leftrightarrow \ln\frac{x^3}{x^2-1} = 0, \quad \frac{x^3}{x^2-1} = 1 \qquad x^3 - x^2 + 1 = 0$

4° Асимптоте

а) В.А. $\lim_{x\to -1+0} \ln\frac{x^3}{x^2-1} = \ln\frac{(-1)^3}{(-1+0)^2-1} = \ln\frac{-1}{1-0-1} = \ln +\infty = +\infty$

$\lim_{x\to 0-} \ln\frac{x^3}{x^2-1} = \ln\frac{0-}{0-1} = \ln\frac{0-}{-1} = \ln 0_+ = -\infty$

$\lim_{x\to 1+0} \ln\frac{x^3}{x^2-1} = \ln\frac{1}{(1+0)^2-1} = \ln\frac{1}{1+0-1} = \ln +\infty = +\infty$

$x=-1$ са десне стране
$x=0$ са леве стране су ВА
$x=1$ са десне стране

б) Х.А. $\lim_{x\to +\infty} \ln\frac{x^3}{x^2-1} = +\infty$ ↯

в) К.А. $y = kx+m$

$k = \lim_{x\to +\infty} \frac{f(x)}{x} = \lim_{x\to +\infty} \frac{\ln\frac{x^3}{x^2-1}}{x} \overset{\text{Л.П.}}{=} \lim_{x\to +\infty} \frac{\frac{x^2-1}{x^3}\cdot\frac{3x^2(x^2-1)-x^3\cdot 2x}{(x^2-1)^2}}{1} = 0$ ↯

5° Екстремне вредности и интервали монотоности

$f'(x) = \dots = \frac{x^2-3}{x(x^2-1)}$

$f'(x) = 0 \Leftrightarrow x^2-3=0$

$\cancel{x=-\sqrt{3}},\ x=\sqrt{3}$

$f_{MIN} = f(\sqrt{3}) = \ln\frac{3\sqrt{3}}{3-1} = \ln\frac{3\sqrt{3}}{2}$

$M\left(\sqrt{3}, \ln\frac{3\sqrt{3}}{2}\right)$

	-1		0		1		$\sqrt{3}$	
x^2-3		−				−	0	+
x		−				+		+
x^2-1		−			0	+		+
$f'(x)$		−	Н.Д.		Н.Д.	−	0	+
$f(x)$	Н.Д.	↘	Н.Д.		Н.Д.	↘	$\ln\frac{3\sqrt{3}}{2}$	↗

Болцано-Кошијева теорема:
Нека је f непрекидна и монотона ф-ја на неком интервалу (a,b). Ако је $f(a)\cdot f(b)<0$ тада $\exists x_0 \in (a,b)$ тако да је $f(x_0)=0$.

3° (наставак)

а) $x\in(-1,0)$: f непрекидна и $f \searrow$

$$\lim_{x\to -1+0} f(x)=+\infty \quad \lim_{x\to 0-} f(x)=-\infty \overset{Б.К.Т}{\Longrightarrow} \exists x_1 \in(-1,0) \quad f(x_1)=0$$

$f(x)>0$ за $x\in(-1,x_1)$
$f(x)<0$ за $x\in(x_1,0)$

б) $x\in(1,\sqrt3)$: f непрекидна и $f \searrow$

$$\lim_{x\to 1+0} f(x)=+\infty \quad f(\sqrt3)=\ln\frac{3\sqrt3}{2}>0 \Rightarrow f(x)>0$$

в) $x\in[\sqrt3,+\infty)$: f непрекидна и $f \nearrow$

$$f(\sqrt3)>0 \quad \lim_{x\to+\infty} f(x)=+\infty \Rightarrow f(x)>0$$

Дакле, ф-ја $f(x)$ има једну нулу $x_1\in(1,0)$
$f(x)>0$ за $x\in(-1,x_1)\cup(1,+\infty)$
$f(x)<0$ за $x\in(x_1,0)$

6° Превојне тачке и интервали $\cup$ и $\cap$

$$f''(x)=\dots=\frac{-x^4+8x^2-3}{x^2(x^2-1)^2}$$

$$f''(x)=0 \Leftrightarrow -x^4+8x^2-3=0 \;/\cdot(-1)$$

смена $x^2=t$: $t^2-8t+3=0$

$t_1=4-\sqrt{13}>0$ $\quad$ $t_2=4+\sqrt{13}>0$

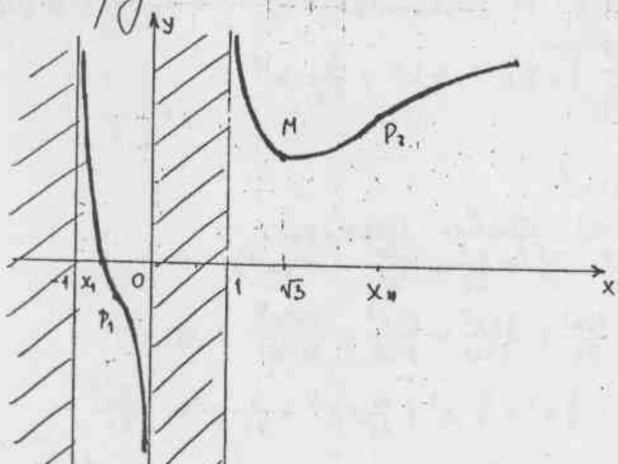
$P_1(x_1, f(x_1))$ $\quad$ $x^2=4-\sqrt{13}$ $\quad$ $x^2=4+\sqrt{13}$

$P_2(x_2, f(x_2))$ $\quad$ $x_1=-\sqrt{4-\sqrt{13}}$ $\quad$ $\cancel{x_3=-\sqrt{4+\sqrt{13}}}$

$\cancel{x_2=\sqrt{4-\sqrt{13}}}$ $\quad$ $x_4=\sqrt{4+\sqrt{13}}$

	-1		x_1		0		1		x_4	
$-x^4+8x^2-3$		$+$	0	$-$				$+$	0	$-$
$f''(x)$	н.д.	$+$	0	$-$	н.д.		н.д.	$+$	0	$-$
$f(x)$	н.д.	$\cup$	$f(x_1)$ п.т.		н.д.		н.д.	$\cup$	$f(x_4)$ п.т.	$\cap$

7° График ф-је



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Маклоренов и Тејлоров полином

$f(x),\ x_0:\quad T_m(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(m)}(x_0)}{}$

$x_0=0:\quad M_m(x) = f(0) + \frac{f'(0)}{1!}\cdot x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(m)}(0)}{m!}x^m$

Зад. 1. $f(x) = \sqrt[3]{x-1}$ $\qquad M_4(x) = ?$

Решење: $M_4(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{IV}(0)}{4!}\cdot x^4$

$f'(x) = \frac{1}{3}(x-1)^{-\frac{2}{3}}$ $\qquad f'(0) = -\frac{1}{3}$

$f''(x) = -\frac{2}{9}(x-1)^{-\frac{5}{3}}$ $\qquad f''(0) = \frac{2}{9}$

$f'''(x) = \frac{10}{27}(x-1)^{-\frac{8}{3}}$ $\qquad f'''(0) = -\frac{10}{27}$

$f^{IV}(x) = -\frac{80}{81}(x-1)^{-\frac{11}{3}}$ $\qquad f^{IV}(0) = \frac{80}{81}$

$M_4(x) = -1 - \frac{1}{3}x + \frac{1}{9}x^2 - \frac{10}{27}\cdot\frac{1}{6}\cdot x^3 + \frac{80}{81}\cdot\frac{1}{24}x^4$

$M_4(x) = -1 - \frac{1}{3}x + \frac{1}{9}x^2 - \frac{5}{81}x^3 + \frac{3'}{443}x^4$

Зад. 2. $f(x) = x\cdot\cos x$ $\qquad M_3(x) = ?$

Решење: $f'(x) = \cos x - x\cdot\sin x$ $\qquad f(0)=0$, $f'(0)=1$

$f''(x) = -\sin x - \sin x - x\cos x = -2\sin x - x\cos x$ $\qquad f''(0)=0$

$f'''(x) = -2\cos x - \cos x + x\sin x = -3\cos x + \sin x$ $\qquad f'''(0)=-3$

$M_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = x - \frac{1}{2}x^3$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + o(x^7)$ $\qquad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + o(x^3)$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + o(x^6)$ $\qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)$

Зад. 3. $f(x) = \underline{x}\cdot\underbrace{\sin 2x}_{M_5(x)}$ $\qquad M_6(x) = ?$

Решење: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5) \Rightarrow \sin 2x = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + o(x^5)$

$M_6(x) = x\cdot\left(2x - \frac{8x^3}{6} + \frac{32x^5}{120}\right) = 2x^2 - \frac{4}{3}x^4 + \frac{4}{15}x^6$

Зад. 4. $f(x) = (x+1)\cdot\underbrace{e^{2x}}_{M_7(x)}$ $\qquad M_7(x) = ?$

$(x+\alpha)$ прави проблем

Решење: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + o(x^7)$

$e^{2x} = 1 + \frac{2x}{1!} + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{24} + \frac{32x^5}{120} + \frac{64x^6}{720} + \frac{128x^7}{4040} + o(x^7)$

$M_7(x) = (1+x)\left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{4}{45}x^6 + \frac{(2x)^7}{7!}\right)$

$= x + 2x^2 + \cdots + \frac{4}{45}x^7 + \left(1 + 2x + 2x^2 + \cdots + \frac{4}{45}x^6 + \frac{2^7x^7}{7!}\right)$

Зад. 5. $f(x)=\frac{1}{\sqrt{3x+1}}$. $x_0=5$ $T_2(x)=?$

Решење: $T_2(x)=f(5)+\frac{f'(5)}{1!}(x-5)+\frac{f''(5)}{2!}(x-5)^2$

$$f(x)=(3x+1)^{-\frac{1}{2}} \qquad f(5)=\frac{1}{4}$$
$$f'(x)=-\frac{1}{2}(3x+1)^{-\frac{3}{2}} \qquad f'(5)=-\frac{3}{2}\cdot\frac{1}{4^3}$$
$$f''(x)=-\frac{3}{2}\cdot\left(-\frac{1}{2}\right)\cdot(3x+1)^{-\frac{5}{2}}\cdot 3 \qquad f''(5)=\frac{27}{4}\cdot\frac{1}{4^3}$$

$$T_2(x)=\frac{1}{4}-\frac{3}{2\cdot 4^3}(x-5)+\frac{27}{8\cdot 4^3}(x-5)^2$$

Зад. 6. $f(x)=\frac{\ln x}{x}$ $x_0=e^2$ $T_3(x)=?$

Решење: $f'(x)=\frac{\frac{1}{x}\cdot x-\ln x}{x^2}=\frac{1-\ln x}{x^2}$ $\qquad f(e^2)=\frac{2}{e^2}$

$f''(x)=\frac{-\frac{1}{x}\cdot x^2-(1-\ln x)\cdot 2x}{x^4}=\frac{x(-1+2-2\ln x)}{x^4}=\frac{1-2\ln x}{x^3}$ $\qquad f'(e^2)=-\frac{1}{e^4}$

$f'''(x)=\frac{-\frac{2}{x}\cdot x^3-(1-2\ln x)\cdot 3x^2}{x^6}=\frac{x^2(-2-3+6\ln x)}{x^6}=\frac{6\ln x-5}{x^4}$ $\qquad f''(e^2)=-\frac{3}{e^6}$

$\qquad f'''(e^2)=\frac{7}{e^8}$

$$T_3(x)=\frac{2}{e^2}-\frac{1}{e^4}(x-e^2)-\frac{3}{2e^6}(x-e^2)^2+\frac{7}{6\cdot e^8}(x-e^2)^3$$

вежбе бр. 15 27. 03. 2007.

Пробна грешке

Зад. 1. $f(x)=\frac{1+\ln|x|}{x}$

Решење: 1° Област дефинисаности

$|x|>0$, $x\neq 0$ $\mathcal{D}_f=(-\infty,0)\cup(0,+\infty)$

2° Парност ф-је

$f(-x)=-f(x)$ непарна

Даље испитујемо ф-ју на $\mathcal{D}_f=(0,+\infty)$ $f(x)=\frac{1+\ln x}{x}$

3° Нуле и знак ф-је

$f(x)=0 \Leftrightarrow 1+\ln x=0$, $x=e^{-1}$, $A\left(\frac{1}{e},0\right)$

4° Асимптоте

а) В.А. $\lim\limits_{x\to 0+}\frac{1+\ln x}{x}=\frac{1+\infty\cdot(-1)}{0+}=\frac{-\infty}{0+}=-\infty$ права $x=0$ је ВА

б) Х.А. $\lim\limits_{x\to+\infty}\frac{1+\ln x}{x}=\lim\limits_{x\to+\infty}\frac{\frac{1}{x}}{1}=0_+$ права $y=0$ је ХА

5° Монотоност и екстремне вредности

$f'(x)=\frac{\frac{1}{x}\cdot x-(1+\ln x)}{x^2}=\frac{-\ln x}{x^2>0}$

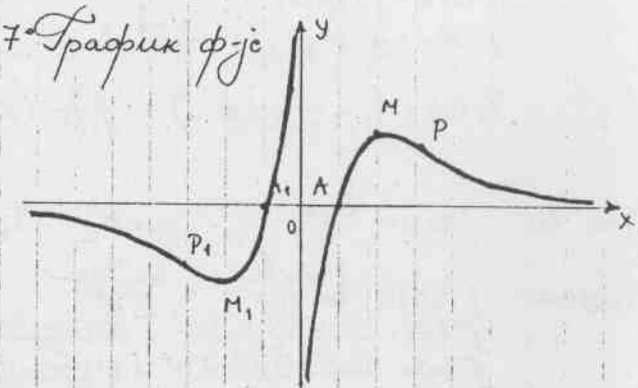
$f'(x)=0 \Leftrightarrow \ln x=0$, $x=1$, $M(1,1)_{max}$

6° Превојне тачке и интервали $\cup$ и $\cap$

$$f''(x) = -\frac{\frac{1}{x}\cdot x^2 - \ln x\cdot 2x}{x^4} = -\frac{x(1-2\ln x)}{x^4} = \frac{2\ln x - 1}{x^3}$$

$f''(x)=0 \qquad P\left(\sqrt{e}, \frac{3}{2}e^{-\frac{1}{2}}\right)$

7° График ф-је



	0		$\sqrt{e}$	
$2\ln x-1$		$-$		$+$
x^3		$+$		$+$
$f''(x)$		$-$		$+$
$f(x)$		$\cap$		$\cup$

Зад.2. $f(x) = \operatorname{arctg}\frac{2x}{1-x^2}$

Решење: 1° Област дефинисаности

$D_f = (-\infty,-1)\cup(-1,1)\cup(1,+\infty)$, $\quad 1-x^2\neq 0$

2° Парност ф-је

$f(-x) = \operatorname{arctg}\frac{-2x}{1-x^2} = -\operatorname{arctg}\frac{2x}{1-x^2} = -f(x) \Rightarrow$ ф-ја је непарна

Испитујемо ф-ју на $D_f = [0,-1)\cup(1,+\infty)$

3° Нуле и знак функције

$f(x)=0 \Leftrightarrow x=0 \qquad A(0,0)$

	0		1	
$2x$		$+$		$+$
$1-x^2$		$+$		$-$
$f(x)$		$+$		$-$

4° Асимптоте

а) В.А. $\lim\limits_{x\to 1-0}\operatorname{arctg}\frac{2}{1-(1-0)} = \operatorname{arctg}\frac{2}{1-1+0} = \operatorname{arctg}(+\infty) = \frac{\pi}{2}$

$\lim\limits_{x\to 1+0}\operatorname{arctg}\frac{2}{1-(1+0)} = \operatorname{arctg}\frac{2}{1-1-0} = \operatorname{arctg}(-\infty) = -\frac{\pi}{2}$

б) Х.А. $\lim\limits_{x\to+\infty}\operatorname{arctg}\frac{2x}{1-x^2} = \operatorname{arctg} 0 = 0$

5° Монотоност и екстремне вредности

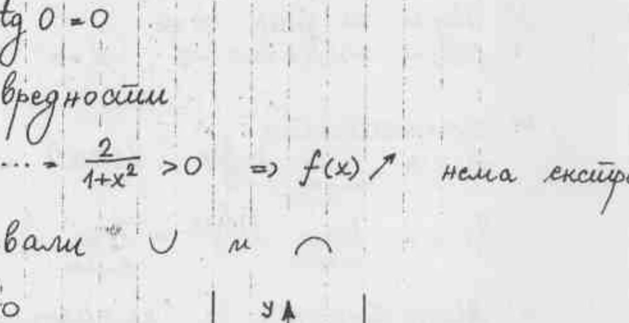
$f'(x) = \frac{1}{1+\left(\frac{2x}{1-x}\right)^2}\cdot\frac{2(1-x^2)-2x(-2x)}{(1-x^2)^2} = \dots = \frac{2}{1+x^2} > 0 \Rightarrow f(x)\nearrow$ нема екстрема

6° Превојне тачке и интервали $\cup$ и $\cap$

$f''(x) = \frac{-2}{(1+x^2)^2}\cdot 2x = \frac{-4x}{(1+x^2)^2} \lessgtr 0$

$P(0,0)$

7° График ф-је



$f'(1) = \frac{2}{1+1} = 1$

$\operatorname{tg}\alpha = 1 \Rightarrow \alpha = \frac{\pi}{4}$

Зад. 5. $f(x)=\frac{1}{\sqrt{3x+1}}$ $x_0=5$ $T_2(x)=?$

Решење: $T_2(x)=f(5)+\frac{f'(5)}{1!}(x-5)+\frac{f''(5)}{2!}(x-5)^2$

$f(x)=(3x+1)^{-\frac{1}{2}}$ $f(5)=\frac{1}{4}$

$f'(x)=-\frac{1}{2}(3x+1)^{-\frac{3}{2}}$ $f'(5)=-\frac{3}{2}\cdot\frac{1}{4^3}$

$f''(x)=-\frac{3}{2}\cdot(-\frac{1}{2})\cdot(3x+1)^{-\frac{5}{2}}\cdot 3$ $f''(5)=\frac{27}{4}\cdot\frac{1}{4^3}$

$T_2(x)=\frac{1}{4}-\frac{3}{2\cdot 4^3}(x-5)+\frac{27}{8\cdot 4^3}(x-5)^2$

Зад. 6. $f(x)=\frac{\ln x}{x}$ $x_0=e^2$ $T_3(x)=?$

Решење: $f'(x)=\frac{\frac{1}{x}\cdot x-\ln x}{x^2}=\frac{1-\ln x}{x^2}$

$f''(x)=\frac{-\frac{1}{x}\cdot x^2-(1-\ln x)\cdot 2x}{x^4}=\frac{x(-1-2+2\ln x)}{x^4}=\frac{-3+2\ln x}{x^3}$

$f'''(x)=\frac{\frac{2}{x}\cdot x^3-(-3+2\ln x)\cdot 3x^2}{x^6}=\frac{x^2(2+9-6\ln x)}{x^6}=\frac{11-6\ln x}{x^4}$

$f(e^2)=\frac{2}{e^2}$

$f'(e^2)=-\frac{1}{e^4}$

$f''(e^2)=\frac{1}{e^6}$

$f'''(e^2)=-\frac{1}{e^8}$

$T_3(x)=\frac{2}{e^2}-\frac{1}{e^4}(x-e^2)+\frac{1}{2e^6}(x-e^2)^2-\frac{1}{6\cdot e^8}(x-e^2)^3$

вежбе бр. 15 27. 03. 2007.

Процена грешке

Зад. 1. $f(x)=\frac{1+\ln|x|}{x}$

Решење: 1° Област дефинисаности

$|x|>0$, $x\neq 0$ $\mathcal{D}_f=(-\infty,0)\cup(0,+\infty)$

2° Парност ф-је

$f(-x)=-f(x)$ непарна

Даље испитујемо ф-ју на $\mathcal{D}_f=(0,+\infty)$ $f(x)=\frac{1+\ln x}{x}$

3° Нуле и знак ф-је

$f(x)=0 \Leftrightarrow 1+\ln x=0$, $x=e^{-1}$, $A(\frac{1}{e},0)$

4° Асимптоте

а) В.А. $\lim\limits_{x\to 0+}\frac{1+\ln x}{x}=\frac{1+\infty\cdot(-1)}{0+}=\frac{-\infty}{0+}=-\infty$ права $x=0$ је ВА

б) Х.А. $\lim\limits_{x\to+\infty}\frac{1+\ln x}{x}=\lim\limits_{x\to+\infty}\frac{\frac{1}{x}}{1}=0_+$ права $y=0$ је ХА

5° Монотоност и екстремне вредности

$f'(x)=\frac{\frac{1}{x}\cdot x-(1+\ln x)}{x^2}=\frac{-\ln x}{x^2}$ $x^2>0$

$f'(x)=0 \Leftrightarrow \ln x=0$, $x=1$, $M(1,1)_{max}$

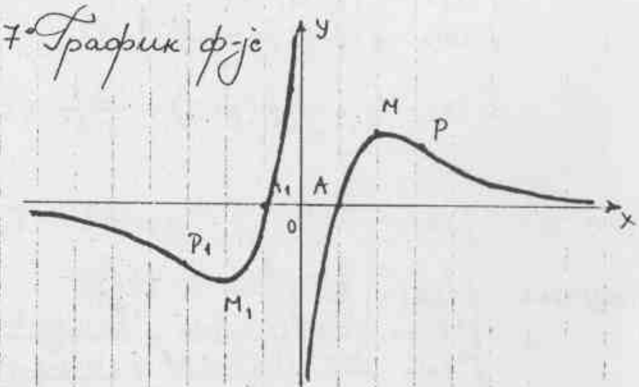
6° Превојне тачке и интервали $\cup$ и $\cap$

$$f''(x) = -\frac{\frac{1}{x}\cdot x^2 - \ln x \cdot 2x}{x^4} = -\frac{x(1-2\ln x)}{x^4} = \frac{2\ln x - 1}{x^3}$$

$f''(x) = 0 \qquad P\left(\sqrt{e}, \frac{3}{2}e^{-\frac{1}{2}}\right)$

	0		$\sqrt{e}$	
$2\ln x - 1$		$-$		$+$
x^3		$+$		$+$
$f''(x)$		$-$		$+$
$f(x)$		$\cap$		$\cup$

7° График ф-је



Зад. 2. $f(x) = \operatorname{arctg} \frac{2x}{1-x^2}$

Решење: 1° Област дефинисаности

$\mathcal{D}_f = (-\infty, -1) \cup (-1, 1) \cup (1, +\infty) \quad , \ 1 - x^2 \neq 0$

2° Парност ф-је

$f(-x) = \operatorname{arctg} \frac{-2x}{1-x^2} = -\operatorname{arctg} \frac{2x}{1-x^2} = -f(x) \Rightarrow$ ф-ја је непарна

Испитујемо ф-ју на $\mathcal{D}_f = [0, -1) \cup (1, +\infty)$

3° Нуле и знак функције

$f(x) = 0 \Leftrightarrow x = 0 \qquad A(0,0)$

	0		1	
$2x$		$+$		$+$
$1-x^2$		$+$		$-$
$f(x)$		$+$		$-$

4° Асимптоте

а) В.А. $\lim\limits_{x\to 1-0} \operatorname{arctg} \frac{2}{1-(1-0)} = \operatorname{arctg} \frac{2}{1-1+0} = \operatorname{arctg}(+\infty) = \frac{\pi}{2}$

$\lim\limits_{x\to 1+0} \operatorname{arctg} \frac{2}{1-(1+0)} = \operatorname{arctg} \frac{2}{1-1-0} = \operatorname{arctg}(-\infty) = -\frac{\pi}{2}$

б) Х.А. $\lim\limits_{x\to +\infty} \operatorname{arctg} \frac{2x}{1-x^2} = \operatorname{arctg} 0 = 0$

5° Монотоност и екстремне вредности

$f'(x) = \frac{1}{1+\left(\frac{2x}{1-x}\right)^2} \cdot \frac{2(1-x^2) - 2x(-2x)}{(1-x^2)^2} = \dots = \frac{2}{1+x^2} > 0 \Rightarrow f(x) \nearrow$ нема екстрема

6° Превојне тачке и интервали $\cup$ и $\cap$

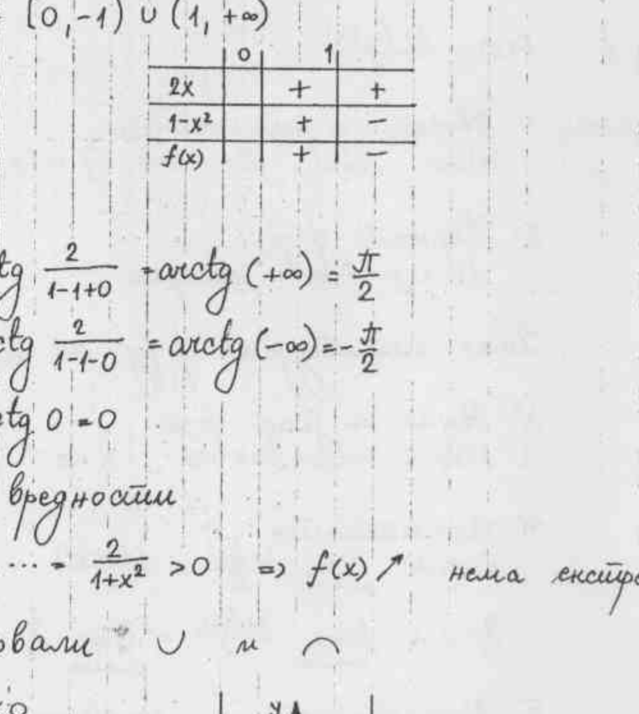
$f''(x) = \frac{-2}{(1+x^2)^2} \cdot 2x = \frac{-4x}{(1+x^2)^2} \leq 0$

$P(0,0)$

7° График ф-је

$f'(1) = \frac{2}{1+1} = 1$

$\operatorname{tg}\alpha = 1 \Rightarrow \alpha = \frac{\pi}{4}$



$f(x) = T_m(x) + R_m(x)$

$$T_m(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(m)}(x_0)}{m!}(x-x_0)^m$$

1° $R_m(x) = \frac{f^{(m+1)}(\xi)}{(m+1)!}(x-x_0)^{m+1}$ Лагранжов облик $\xi = x_0 + \theta(x-x_0)$, $0<\theta<1$

2° $R_m(x) = o\left((x-x_0)^m\right)$ Пеанов облик

Зад. 1. Ф-ју $f(x) = \sqrt[3]{1+x}$ апроксимирати Маклореновим полиномом 3. степена, а затим користећи добијену формулу израчунати приближно $\sqrt[3]{1{,}3}$ и проценити грешку

Решење: $f(0) = 1$

$f'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}}$ $\quad f'(0) = \frac{1}{3}$

$f''(x) = -\frac{2}{9}(1+x)^{-\frac{5}{3}}$ $\quad f''(0) = -\frac{2}{9}$

$f'''(x) = \frac{10}{27}(1+x)^{-\frac{8}{3}}$ $\quad f'''(0) = \frac{10}{27}$

$f^{IV}(x) = -\frac{80}{81}(1+x)^{-\frac{11}{3}}$ $\quad f^{IV}(0) = -\frac{80}{81}$

1) $|a\cdot b| = |a|\cdot|b|$
2) $|a\pm b| \le |a| \pm |b|$
3) $|a^m| \le |a|^m$

$f(x) = M_3(x) + R_3(x)$

$M_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

$M_3(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3$

$f(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 + R_3(x)$

$$\sqrt[3]{1{,}3} = \sqrt[3]{1+0{,}3} = f(0{,}3) \approx 1 + \frac{1}{3}\cdot 0{,}3 - \frac{1}{9}\cdot 0{,}3^2 + \frac{5}{81}\cdot 0{,}3^3 = 1{,}0917$$

$$|R_3(x)| = \left|\frac{f^{IV}(\theta x)}{4!}\cdot x^4\right| = \left|-\frac{80}{81(1+\theta x)^{\frac{11}{3}}}\cdot\frac{x^4}{4!}\right| \le \frac{80}{81\cdot 4!}\cdot\frac{1}{|1+\theta x|}\cdot x^4$$

$$\le \frac{80}{81\cdot 4!}\left(\frac{3}{10}\right)^4 = \frac{1}{3}\cdot 10^{-3} = 0{,}0003$$

$\sqrt[3]{1{,}3} = 1{,}09139$ грешка на 4. децимали

Зад. 2. Функцију $f(x) = \operatorname{arctg}\frac{x-1}{x+1}$ апроксимирати Маклореновим полиномом 4. степена и оценити грешку апроксимације на $x \in [0, \frac{1}{10}]$

Решење: $f(x) = \operatorname{arctg}\frac{x-1}{x+1}$ $\quad f(0) = -\frac{\pi}{4}$

$f'(x) = \frac{1}{x^2+1}$ $\quad f'(0) = 1$

$f''(x) = -\frac{2x}{(x^2+1)^2}$ $\quad f''(0) = 0$

$f'''(x) = \frac{2(3x^2-1)}{(x^2+1)^3}$ $\quad f'''(0) = -2$

$f^{IV}(x) = \frac{24(1-x^2)\cdot x^2}{(x^2+1)^5}$ $\quad f^{IV}(0) = 0$

$f(x) = M_4(x) + R_4(x)$

$f(x) = -\frac{\pi}{4} + x - \frac{1}{3}x^3 + R_4(x)$

$$|R_4(x)| = \left|\frac{f^{V}(\theta x)}{5!}\cdot x^5\right| = \left|\frac{24(5\cdot(\theta x)^4 - 10(\theta x)^2 + 1)}{120((\theta x)^2+1)^5}\cdot x^5\right| \le \frac{5|\theta x|^4 + 10|\theta x|^2 + 1}{5\cdot|(\theta x)^2+1|^5}\cdot|x|^5$$

$$\le \frac{5\cdot 10^{-4} + 10^{-1} + 1}{5}\cdot 10^{-5} = \dots = \frac{11005}{5}\cdot 10^{-9} = 2201\cdot 10^{-9} \le 2500\cdot 10^{-9} = \frac{1}{4}\cdot 10^{-5}$$

$|R_4(x)| \le \frac{1}{4}\cdot 10^{-5}$

$x \in (0, \frac{1}{10})$

$0 < \theta < 1$

$0 < \theta x < \frac{1}{10}$

Зад. 3. Апроксимирати ф-ју $f(x)=\frac{\ln(1+x)}{\sqrt{1+x}}$ Маклореновим полиномом 3. степена на интервалу $x\in[-\frac{1}{4},\frac{1}{4}]$

Решење: $f(x)= x-x^2+\frac{23}{24}x^3+R_3(x)$

$$f^{IV}(x)=\frac{\frac{105}{16}\cdot\ln(1+x)-22}{(1+x)^{9/2}}$$

$$|R_3(x)|=\left|\frac{f^{IV}(\theta x)}{4!}\cdot x^4\right|=\left|\frac{\frac{105}{16}\cdot\ln(1+\theta x)-22}{24\cdot(1+\theta x)^{9/2}}\cdot x^4\right|\le\frac{1}{24}\cdot\frac{\frac{105}{16}|\ln(1+\theta x)|+22}{|1+\theta x|^{9/2}}\cdot|x|^4$$

$$\le\frac{1}{24}\cdot\frac{\frac{105}{16}\cdot\ln\frac{5}{4}+22}{(\frac{3}{4})^{\frac{9}{2}}}\cdot\left(\frac{1}{4}\right)^4\le\frac{24}{24\cdot(\frac{3}{4})^{\frac{9}{2}}}\cdot\frac{1}{4^4}=\frac{2}{81\sqrt{3}}<0,015$$

$-\frac{1}{4}\le x\le\frac{1}{4}$

$0<\theta<1$

$-\frac{1}{4}<\theta x<\frac{1}{4}$

$\frac{3}{4}<1+\theta x<\frac{5}{4}$

$\ln\frac{3}{4}<\ln(1+\theta x)<\ln\frac{5}{4}$

вежба бр. 16 — 24. 04. 2007.

Неодређени интеграли

Подсетимо се најпре таблице извода:

1° $(c)'=0$
2° $(x^\alpha)'=\alpha\cdot x^{\alpha-1}$
3° $(e^x)'=e^x$
4° $(a^x)'=a^x\cdot\ln a$
5° $(\ln x)'=\frac{1}{x}$
6° $(\log_a x)'=\frac{1}{x\cdot\ln a}$
7° $(\sin x)'=\cos x$
8° $(\cos x)'=-\sin x$
9° $(tg x)'=\frac{1}{\cos^2 x}$
10° $(ctg x)'=-\frac{1}{\sin^2 x}$
11° $(\arcsin x)'=\frac{1}{\sqrt{1-x^2}}$
12° $(\arccos x)'=-\frac{1}{\sqrt{1-x^2}}$
13° $(arctg x)'=\frac{1}{1+x^2}$
14° $(arcctg x)'=-\frac{1}{1+x^2}$
15° $(sh x)'=ch x$
16° $(ch x)'=sh x$
17° $(th x)'=\frac{1}{ch^2 x}$
18° $(cth x)'=\frac{1}{sh^2 x}$

Таблица интеграла:

1° $\int dx = x + C$
2° $\int x^\alpha\cdot dx=\frac{x^{\alpha+1}}{\alpha+1}+C$
3° $\int\frac{dx}{x}=\ln|x|+C$
4° $\int a^x dx=\frac{a^x}{\ln a}+C$
5° $\int e^x dx=e^x+C$
6° $\int\sin x\,dx=-\cos x+C$
7° $\int\cos x\,dx=\sin x+C$
8° $\int\frac{dx}{\cos^2 x}=tg x+C$
9° $\int\frac{dx}{\sin^2 x}=-ctg x+C$
10° $\int\frac{dx}{1+x^2}=arctg x+C$
11° $\int\frac{dx}{\sqrt{1-x^2}}=\arcsin x+C$
12° $\int\frac{dx}{\sqrt{x^2\pm a}}=\ln|x+\sqrt{x^2\pm a}|+C$
13° $\int\frac{dx}{a-x^2}=\frac{1}{2\sqrt{a}}\ln\left|\frac{\sqrt{a}+x}{\sqrt{a}-x}\right|+C$
14° $\int sh x\,dx=ch x$, $sh x=\frac{e^x-e^{-x}}{2}$
15° $\int ch x\,dx=sh x$, $ch x=\frac{e^x+e^{-x}}{2}$

Правила интеграције:

1° $\int(f(x)+g(x))\,dx=\int f(x)dx+\int g(x)dx$

2° $\int a f(x)\,dx=a\int f(x)dx$

3° $\left(\int f(x)\,dx\right)'=f(x)$

4° $\int f'(x)\,dx=f(x)+C$

I) Таблични

Зад. 1. $\int \frac{\sqrt{x}-2\sqrt[3]{x}+1}{\sqrt{x}}\,dx \Leftrightarrow *$

Решење: $* = \int\left(1-2\cdot x^{\frac{1}{3}-\frac{1}{2}}+x^{-\frac{1}{2}}\right)dx = \int dx - 2\int x^{-\frac{1}{6}}dx + \int x^{-\frac{1}{2}}dx$

$= x - 2\cdot\frac{x^{\frac{5}{6}}}{\frac{5}{6}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = x - \frac{12}{5}\cdot x^{5/6} + 2x^{\frac{1}{2}} + C$

Зад. 2. $\int\left(1-\frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\,dx \Leftrightarrow *$

Решење: $* = \int\left(1-\frac{1}{x^2}\right)\sqrt{x^{\frac{3}{2}}} = \int\left(1-\frac{1}{x^2}\right)x^{\frac{3}{4}}dx = \int x^{\frac{3}{4}}dx - \int x^{-\frac{5}{4}}dx = \frac{x^{\frac{7}{4}}}{\frac{7}{4}} - \frac{x^{-\frac{1}{4}}}{-\frac{1}{4}} = \frac{7}{4}x^{\frac{7}{4}} + 4x^{-\frac{1}{4}} + C$

Зад. 3. $\int\frac{x^2+3}{x^2-1}dx \Leftrightarrow *$

Решење: $* = \int\frac{x^2-1+4}{x^2-1}dx = \int dx + 4\int\frac{dx}{x^2-1} = x + 4\cdot\frac{1}{2}\cdot\ln\left|\frac{x-1}{x+1}\right| + C = x + 2\cdot\ln\left|\frac{x-1}{x+1}\right| + C$

Зад. 4. $\int\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1-x^4}}dx \Leftrightarrow *$

Решење: $* = \int\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1-x^2}\cdot\sqrt{1+x^2}}dx = \int\frac{dx}{\sqrt{1-x^2}} - \int\frac{dx}{\sqrt{1+x^2}} = \arcsin x - \ln\left|x+\sqrt{1+x^2}\right|$

Зад. 5. $\int\frac{e^{3x}+1}{e^x+1}dx \Leftrightarrow *$

Решење: $* = \int\frac{(e^x+1)(e^{2x}-e^x+1)}{e^x+1}dx = \int e^{2x}dx - \int e^x dx + \int dx = \frac{1}{2}e^{2x} - e^x + x + C$

Зад. 6. $\int\frac{\cos 2x}{\cos^2x\cdot\sin^2x}dx \Leftrightarrow *$

Решење: $* = \int\frac{\cos^2x-\sin^2x}{\cos^2x\cdot\sin^2x}dx = \int\frac{dx}{\sin^2x} - \int\frac{dx}{\cos^2x} = -\operatorname{ctg}x - \operatorname{tg}x + C$

Зад. 7. $\int \operatorname{tg}^2x\cdot dx \Leftrightarrow *$

Решење: $* = \int\frac{\sin^2x}{\cos^2x}dx = \int\frac{1-\cos^2x}{\cos^2x}dx = \int\frac{dx}{\cos^2x} - \int dx = \operatorname{tg}x - x + C$

II) Смена променливе

Зад. 1. $\int\frac{x^4\,dx}{\sqrt{4+x^5}} \Leftrightarrow *$

Решење: $* = \left[\begin{array}{l}\text{смена:}\\ 4+x^5=t\\ 5x^4\,dx = 1\cdot dt\\ x^4dx = \frac{dt}{5}\end{array}\right] = \int\frac{\frac{dt}{5}}{\sqrt{t}} = \frac{1}{5}\int t^{-\frac{1}{2}}dt = \frac{1}{5}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{5}\cdot\sqrt{4+x^5} + C$

Зад. 2. $\int e^x\cdot\sin e^x\cdot dx \Leftrightarrow *$

Решење: $* = \left[\begin{array}{l}\text{смена: } e^x=t\\ e^x dx = dt\end{array}\right] = \int\sin t\,dt = -\cos t + C = -\cos e^x + C$

Зад. 3. $\int \frac{e^{2x}dx}{\sqrt{e^x-1}} \Leftrightarrow *$

Решење: $* = \int \frac{t+1}{\sqrt{t}}\,dt = \int \sqrt{t}\cdot dt + \int \frac{dt}{\sqrt{t}} = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{3}(e^x-1)^{\frac{3}{2}} + 2(e^x-1)^{\frac{1}{2}} + C$

Зад. 4. $\int \frac{\sqrt{x}}{x(1+\sqrt[3]{x})}\,dx \Leftrightarrow *$

Решење: $* = \begin{bmatrix} \text{смена: } x = t^6 \\ dx = 6t^5 dt \end{bmatrix} = \int \frac{t^{\cancel{3}2}}{t^{\cancel{6}}(1+t^2)}\cdot 6\cancel{t^5}dt = 6\int \frac{t^2}{1+t^2}dt = 6\int \frac{t^2+1-1}{t^2+1}$

$= 6\int dt - 6\int \frac{1}{t^2+1} = 6(t - \operatorname{arctg} t) + C = 6(\sqrt[6]{x} - \operatorname{arctg}\sqrt[6]{x}) + C$

Зад. 5. $\int \frac{dx}{\sin x} \Leftrightarrow *$

Решење: $* = \int \frac{\sin x}{\sin^2 x}dx = \int \frac{\sin x}{1-\cos^2 x} = \begin{bmatrix} \text{смена: } \cos x = t \\ -\sin x\,dx = dt \end{bmatrix} = -\int \frac{dt}{1-t^2} = -\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + C = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + C$

Зад. 6. $\int \sin^2 x\,dx \Leftrightarrow *$

Решење: $* = \int \frac{1-\cos 2x}{2}dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

$\int \sin(mx)dx = -\frac{\cos mx}{m} + C \qquad \int \cos(mx)dx = \frac{\sin mx}{m} + C$

* Када имамо паран степен у збиру радимо тригонометрију двоструког угла
* Када у збиру непаран увек sinx или cosx остаје горе као вишак

Зад. 7. $\int \sin^2 x \cdot \cos^3 x \cdot dx \Leftrightarrow *$

Решење: $* = \int \sin^2 x \cdot \cos^2 x \cdot \cos x \cdot dx = \int \sin^2 x \cdot (1-\sin^2 x)\cdot \cos x\,dx = \begin{bmatrix} \text{смена: } \sin x = t \\ \cos x\,dx = dt \end{bmatrix}$

$= \int t^2(1-t^2)dt = \int t^2 dt - \int t^4 dt = \frac{t^3}{3} - \frac{t^5}{5} + C = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$

Зад. 8. $\int \sqrt{\frac{e^x-1}{e^x+1}}\,dx \Leftrightarrow *$

Решење: $* = \begin{bmatrix} \text{смена: } e^x = t, & t\,dx = dt \\ e^x dx = dt, & dx = \frac{dt}{t} \end{bmatrix} = \int \sqrt{\frac{t-1}{t+1}}\cdot\frac{dt}{t} = \int \frac{(t-1)dt}{t\sqrt{t^2-1}}$

$= \int \frac{t\,dt}{t\sqrt{t^2-1}} - \int \frac{dt}{t\sqrt{t^2-1}} = \ln\left|t + \sqrt{t^2-1}\right| - I_1$

$I_1 = \int \frac{dt}{t\sqrt{t^2-1}} = \begin{bmatrix} \text{смена: } t = \frac{1}{z} \\ dt = -\frac{dz}{z^2} \end{bmatrix} = \int \frac{-\frac{dz}{z^2}}{\frac{1}{z}\sqrt{\frac{1}{z^2}-1}} = -\int \frac{\frac{dz}{z^2}}{\frac{1}{z}\frac{\sqrt{1-z^2}}{z}} = -\int \frac{dz}{\sqrt{1-z^2}} = \arcsin z + C = -\arcsin\frac{1}{t} + C$

$I = \ln\left|e^x + \sqrt{e^{2x}-1}\right| + \arcsin e^{-x} + C$

I) Таблични

Зад. 1. $\int \frac{\sqrt{x} - 2\sqrt[3]{x}+1}{\sqrt{x}}\,dx \Leftrightarrow *$

Решење: $* = \int\left(1 - 2\cdot x^{\frac{1}{3}-\frac{1}{2}} + x^{-\frac{1}{2}}\right)dx = \int dx - 2\int x^{-\frac{1}{6}}dx + \int x^{-\frac{1}{2}}dx$

$= x - 2\cdot\frac{x^{\frac{5}{6}}}{\frac{5}{6}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = x - \frac{12}{5}\cdot x^{5/6} + 2x^{\frac{1}{2}} + C$

Зад. 2. $\int\left(1-\frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\,dx \Leftrightarrow *$

Решење: $* = \int\left(1-\frac{1}{x^2}\right)\sqrt{x^{\frac{3}{2}}} = \int\left(1-\frac{1}{x^2}\right)x^{\frac{3}{4}}dx = \int x^{\frac{3}{4}}dx - \int x^{-\frac{5}{4}}dx = \frac{x^{\frac{7}{4}}}{\frac{7}{4}} - \frac{x^{-\frac{1}{4}}}{-\frac{1}{4}} = \frac{7}{4}x^{\frac{7}{4}} + 4x^{-\frac{1}{4}} + C$

Зад. 3. $\int\frac{x^2+3}{x^2-1}\,dx \Leftrightarrow *$

Решење: $* = \int\frac{x^2-1+4}{x^2-1}\,dx = \int dx + 4\int\frac{dx}{x^2-1} = x + 4\cdot\frac{1}{2}\cdot\ln\left|\frac{x-1}{x+1}\right| + C = x + 2\cdot\ln\left|\frac{x-1}{x+1}\right| + C$

Зад. 4. $\int\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1-x^4}}\,dx \Leftrightarrow *$

Решење: $* = \int\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1-x^2}\cdot\sqrt{1+x^2}}\,dx = \int\frac{dx}{\sqrt{1-x^2}} - \int\frac{dx}{\sqrt{1+x^2}} = \arcsin x - \ln\left|x+\sqrt{1+x^2}\right|$

Зад. 5. $\int\frac{e^{3x}+1}{e^x+1}\,dx \Leftrightarrow *$

Решење: $* = \int\frac{(e^x+1)(e^{2x}-e^x+1)}{e^x+1}\,dx = \int e^{2x}dx - \int e^x dx + \int dx = \frac{1}{2}e^{2x} - e^x + x + C$

Зад. 6. $\int\frac{\cos 2x}{\cos^2 x\cdot\sin^2 x}\,dx \Leftrightarrow *$

Решење: $* = \int\frac{\cos^2 x - \sin^2 x}{\cos^2 x\cdot\sin^2 x}\,dx = \int\frac{dx}{\sin^2 x} - \int\frac{dx}{\cos^2 x} = -\operatorname{ctg} x - \operatorname{tg} x + C$

Зад. 7. $\int \operatorname{tg}^2 x\cdot dx \Leftrightarrow *$

Решење: $* = \int\frac{\sin^2 x}{\cos^2 x}\,dx = \int\frac{1-\cos^2 x}{\cos^2 x}\,dx = \int\frac{dx}{\cos^2 x} - \int dx = \operatorname{tg} x - x + C$

II) Смена променљиве

Зад. 1. $\int\frac{x^4\,dx}{\sqrt{4+x^5}} \Leftrightarrow *$

Решење: $* = \left[\begin{array}{l}\text{смена:}\\ 4+x^5=t\\ 5x^4\,dx = 1\cdot dt\\ x^4dx = \frac{dt}{5}\end{array}\right] = \int\frac{\frac{dt}{5}}{\sqrt{t}} = \frac{1}{5}\int t^{-\frac{1}{2}}dt = \frac{1}{5}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{5}\cdot\sqrt{4+x^5} + C$

Зад. 2. $\int e^x\cdot\sin e^x\cdot dx \Leftrightarrow *$

Решење: $* = \left[\begin{array}{l}\text{смена: } e^x = t\\ \quad e^x dx = dt\end{array}\right] = \int \sin t\,dt = -\cos t + C = -\cos e^x + C$

Зад. 3. $\int \frac{e^{2x}dx}{\sqrt{e^x-1}} \Leftrightarrow *$

Решење: $* = \int \frac{t+1}{\sqrt{t}}\,dt = \int \sqrt{t}\,dt + \int \frac{dt}{\sqrt{t}} = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{3}(e^x-1)^{\frac{3}{2}} + 2(e^x-1)^{\frac{1}{2}} + C$

Зад. 4. $\int \frac{\sqrt{x}}{x(1+\sqrt[3]{x})}\,dx \Leftrightarrow *$

Решење: $* = \left[\begin{matrix} \text{смена: } x=t^6 \\ dx=6t^5dt \end{matrix}\right] = \int \frac{t^{\cancel{3}2}}{\cancel{t^6}(1+t^2)}\cdot 6\cancel{t^5}dt = 6\int \frac{t^2}{1+t^2}dt = 6\int \frac{t^2+1-1}{t^2+1}$

$= 6\int dt - 6\int \frac{1}{t^2+1} = 6(t - \mathrm{arctg}\, t) + C = 6(\sqrt[6]{x} - \mathrm{arctg}\sqrt[6]{x}) + C$

Зад. 5. $\int \frac{dx}{\sin x} \Leftrightarrow *$

Решење: $* = \int \frac{\sin x}{\sin^2 x}dx = \int \frac{\sin x}{1-\cos^2 x} = \left[\begin{matrix} \text{смена: } \cos x = t \\ -\sin x\,dx = dt \end{matrix}\right] = -\int \frac{dt}{1-t^2} = -\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + C = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + C$

Зад. 6. $\int \sin^2 x\,dx \Leftrightarrow *$

Решење: $* = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

$\int \sin(nx)\,dx = -\frac{\cos mx}{n} + C \qquad \int \cos(nx)\,dx = \frac{\sin nx}{n} + C$

* Када имамо паран степен у збиру радимо тригонометрију двоструког угла
* Када у збиру непаран увек (sinx или cosx) остаје горе (као вишак)

Зад. 7. $\int \sin^2 x\cdot\cos^3 x\cdot dx \Leftrightarrow *$

Решење: $* \int \sin^2 x\cdot\cos^2 x\cdot\cos x\cdot dx = \int \sin^2 x\cdot(1-\sin^2 x)\cdot\cos x\,dx = \left[\begin{matrix} \text{смена: } \sin x = t \\ \cos x\,dx = dt \end{matrix}\right]$

$= \int t^2(1-t^2)\,dt = \int t^2dt - \int t^4dt = \frac{t^3}{3} - \frac{t^5}{5} + C = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$

Зад. 8. $\int \sqrt{\frac{e^x-1}{e^x+1}}\,dx \Leftrightarrow *$

Решење: $* = \left[\begin{matrix} \text{смена: } e^x = t & ; & t\,dx = dt \\ e^x dx = dt & ; & dx = \frac{dt}{t} \end{matrix}\right] = \int \sqrt{\frac{t-1}{t+1}}\cdot\frac{dt}{t} = \int \frac{(t-1)\,dt}{t\sqrt{t^2-1}}$

$= \int \frac{\cancel{t}dt}{\cancel{t}\sqrt{t^2-1}} - \int \frac{dt}{t\sqrt{t^2-1}} = \ln|t + \sqrt{t^2-1}| - I_1$

$I_1 = \int \frac{dt}{t\sqrt{t^2-1}} = \left[\begin{matrix} \text{смена: } t = \frac{1}{z} \\ dt = -\frac{dz}{z^2} \end{matrix}\right] = \int \frac{-\frac{dz}{z^2}}{\frac{1}{z}\sqrt{\frac{1}{z^2}-1}} = -\int \frac{\frac{dz}{z^2}}{\frac{1}{z}\frac{\sqrt{1-z^2}}{z}} = -\int \frac{dz}{\sqrt{1-z^2}} = \arcsin z + C = -\arcsin\frac{1}{t} + C$

$I = \ln|e^x + \sqrt{e^{2x}-1}| + \arcsin e^{-x} + C$

Зад. 9. $\int \frac{3x-1}{x^2-x+1}\,dx \Leftrightarrow *$

$$\int \frac{dx}{x^2+a} = \frac{1}{\sqrt{a}}\cdot \operatorname{arctg}\frac{x}{\sqrt{a}} + C$$

Решење: $(x^2-x+1)' = 2x-1$

$$* = 3\int \frac{x-\frac{1}{3}}{x^2-x+1}\,dx = 3\int \frac{\frac{1}{2}(2x-1)+\frac{1}{2}-\frac{1}{3}}{x^2-x+1}\,dx = \frac{3}{2}\int \frac{2x-1}{x^2-x+1}\,dx + \frac{1}{2}\int \frac{dx}{x^2-x+1}$$

$$= \frac{3}{2}\int \frac{d(x^2-x+1)}{x^2-x+1} + \frac{1}{2}\int \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}}$$

$$I_1 = \int \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}} = \left[\begin{array}{l} \text{смена}: x-\frac{1}{2}=t \\ dx=dt \end{array}\right] = \int \frac{dt}{t^2+\frac{3}{4}} = \frac{4}{3}\int \frac{dt}{\frac{4}{3}t^2+1} = \frac{4}{3}\int \frac{dt}{(\frac{2}{\sqrt{3}}t)^2+1}$$

$$= \left[\begin{array}{l} \text{смена}: \\ \frac{2}{\sqrt{3}}t = z \\ \frac{2}{\sqrt{3}}dt = dz \end{array}\right] = \frac{4}{3}\int \frac{\frac{\sqrt{3}}{2}dz}{z^2+1} = \frac{2}{\sqrt{3}}\operatorname{arctg} z + C = \frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}} + C$$

$$I = \frac{3}{2}\ln(x^2-x+1) + \frac{1}{2}\cdot\frac{1}{\sqrt{\frac{3}{4}}}\operatorname{arctg}\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C$$

$$= \frac{3}{2}\ln(x^2-x+1) + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}} + C$$

Зад. 10. $\int \frac{x+3}{\sqrt{4x^2+4x+3}}\,dx \Leftrightarrow *$

Решење: $(4x^2+4x+3)' = 8x+4$

$$* = \int \frac{\frac{1}{8}(8x+4)-\frac{1}{2}+3}{\sqrt{4x^2+4x+3}}\,dx = \frac{1}{8}\int \frac{8x+4}{\sqrt{4x^2+4x+3}}\,dx + \frac{5}{2}\int \frac{dx}{\sqrt{4x^2+4x+3}}$$

$$= \frac{1}{8}\int \frac{d(4x^2+4x+3)}{\sqrt{4x^2+4x+3}} + \frac{5}{2}\int \frac{dx}{\sqrt{(2x+1)^2+2}} = \left[\begin{array}{l} \text{смена}: 2x+1=t \\ 2dx = dt \end{array}\right]$$

$$= \frac{1}{8}\cdot\frac{\sqrt{4x^2+4x+3}}{\frac{1}{2}} + \frac{5}{2}\cdot\frac{1}{2}\int \frac{dt}{\sqrt{t^2+2}} = \frac{1}{4}\sqrt{4x^2+4x+3} + \frac{5}{4}\ln\left|t+\sqrt{t^2+2}\right| + C$$

$$= \frac{1}{4}\sqrt{4x^2+4x+3} + \frac{5}{4}\ln\left|(2x+1)+\sqrt{4x^2+4x+3}\right| + C$$

Зад. 11. $\int \frac{dx}{x\cdot\ln x\cdot(\ln\ln x)^3}$

Решење: $\int \frac{dx}{x\cdot\ln x\cdot(\ln\ln x)^3} = \left[\begin{array}{l} \text{смена}: \ln x = t \\ \frac{dx}{x} = dt \end{array}\right] = \int \frac{dt}{t\cdot(\ln t)^3} = \left[\begin{array}{l} \text{смена}: \ln t = z \\ \frac{dt}{t} = dz \end{array}\right] = \int \frac{dz}{z^3}$

$$= \frac{z^{-2}}{-2} + C = -\frac{1}{2}\cdot\frac{1}{(\ln t)^2} + C = -\frac{1}{2}\cdot\frac{1}{(\ln\ln x)^2} + C$$

могло је и смено $\ln\ln x = t$

Зад. 12. $\int \frac{\ln x\,dx}{x\sqrt{1+\ln x}} \Leftrightarrow *$

Решење: $* = \left[\begin{array}{l} \text{смена}: \ln x = t \\ \frac{dx}{x} = dt \end{array}\right] = \int \frac{t\cdot dt}{\sqrt{1+t}} = \left[\begin{array}{l} \text{смена}: 1+t = z \\ dt = dz \end{array}\right] = \int \frac{(z-1)dz}{\sqrt{z}}$

$$= \int \sqrt{z}\,dz - \int \frac{dz}{\sqrt{z}} = \frac{z^{\frac{3}{2}}}{\frac{3}{2}} - \frac{z^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{3}(1+\ln x)^{\frac{3}{2}} - 2(1+\ln x)^{\frac{1}{2}} + C$$

Парцијална интеграција

$\int u\,dv = u\cdot v - \int v\,du$ (за u се узима $\log x$, $P_n(x)$, $\operatorname{arctg} x$, $\arcsin x$)

Помаже нам за ф-је чије интеграле немамо у таблици.

Зад. 1. $\int x^2 \sin 2x\,dx$

Решење: $\int x^2\cdot \sin 2x\,dx = \begin{bmatrix} \text{п.и.} & \\ u=x^2 & dv=\sin 2x\,dx \\ du=2x\,dx & v=-\frac{\cos 2x}{2} \end{bmatrix} = -\frac{x^2\cdot\cos 2x}{2} + \int \frac{\cos 2x}{\cancel{2}}\cdot\cancel{2}x\,dx$

$$= -\frac{x^2}{2}\cos 2x + \int x\cdot\cos 2x\,dx = \begin{bmatrix} \text{п.и.} & \\ u=x & dv=\cos 2x\,dx \\ du=dx & v=\frac{\sin 2x}{2} \end{bmatrix}$$

$$= -\frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x - \frac{1}{2}\int \sin 2x\,dx = -\frac{x^2}{2}\cos 2x - \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C$$

Зад. 2. $\int x\ln^2 x\,dx \Leftrightarrow *$

Решење: $* = \begin{bmatrix} \text{п.и.} & \\ u=\ln^2 x & dv=x\,dx \\ du=2\ln x\cdot\frac{dx}{x} & v=\frac{x^2}{2} \end{bmatrix} = \frac{x^2}{2}\ln^2 x - \int \frac{x^{\cancel{2}}}{\cancel{2}}\cdot\cancel{2}\ln x\,\frac{dx}{\cancel{x}} = \frac{x^2}{2}\ln^2 x - \int x\ln x\,dx$

$$= \begin{bmatrix} \text{п.и.} & \\ u=\ln x & dv=x\,dx \\ du=\frac{dx}{x} & v=\frac{x^2}{2} \end{bmatrix} = \frac{x^2}{2}\ln^2 x - \frac{x^2}{2}\ln x + \frac{1}{2}\int x\,dx = \frac{x^2}{2}\left(\ln^2 x - \ln x + \frac{1}{2}\right) + C$$

Зад. 3. $\underline{I} = \int e^x\cdot\cos 2x\,dx$

Решење: у овом случају свеједно коју ф-ју узимамо за u

$$I = \begin{bmatrix} u=e^x & dv=\cos 2x\,dx \\ du=e^x dx & v=\frac{1}{2}\sin 2x \end{bmatrix} = \frac{1}{2}e^x\sin 2x - \frac{1}{2}\int e^x\cdot\sin 2x\,dx = \begin{bmatrix} u=e^x & dv=\sin 2x \\ du=e^x dx & v=-\frac{1}{2}\cos 2x \end{bmatrix}$$

$$= \frac{1}{2}e^x\cdot\sin 2x + \frac{1}{4}e^x\cos 2x - \frac{1}{4}\int e^x\cos 2x\,dx$$

$$\underline{I} = \frac{1}{2}e^x\cdot\sin 2x + \frac{1}{4}e^x\cos 2x - \frac{1}{4}I$$

$$\frac{5}{4}I = \frac{1}{2}e^x\cdot\sin 2x + \frac{1}{4}e^x\cos 2x$$

$$\underline{I} = \frac{1}{5}e^x(2\sin 2x + \cos 2x) + C$$

Зад. 4. $\int \frac{x\cdot\ln x}{(1-x^2)\sqrt{x^2-1}}dx \Leftrightarrow *$ $\qquad\qquad \int\frac{dx}{x\sqrt{x^2-1}} = -\arcsin\frac{1}{x} + C$

Решење: $* = \begin{bmatrix} u=\ln x & dv=\frac{x\,dx}{(1-x^2)\sqrt{x^2-1}} \\ du=\frac{dx}{x} & v=-\int\frac{x\,dx}{(x^2-1)^{\frac{3}{2}}} = -\frac{1}{2}\int\frac{d(x^2-1)}{(x^2-1)^{\frac{3}{2}}} = -\frac{1}{2}\,\frac{(x^2-1)^{-\frac{1}{2}}}{-\frac{1}{2}} = \frac{1}{\sqrt{x^2-1}} \end{bmatrix}$

$$= \frac{\ln x}{\sqrt{x^2-1}} - \int\frac{dx}{x\sqrt{x^2-1}}$$

$$I_1 = \int\frac{dx}{x\sqrt{x^2-1}} = \begin{bmatrix} \text{смена: } x=\frac{1}{t} \\ dx=-\frac{dt}{t^2} \end{bmatrix} = \int\frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{\frac{1}{t^2}-1}} = -\int\frac{\frac{dt}{t^2}}{\frac{1}{t}\cdot\frac{\sqrt{1-t^2}}{t}} = -\int\frac{dt}{\sqrt{1-t^2}} = -\arcsin t + C$$

$$I = \frac{\ln x}{\sqrt{x^2-1}} + \arcsin\frac{1}{x} + C$$

Зад. 5. $\int (\arcsin x)^2 dx \iff *$

Решење: $* = \begin{bmatrix} u=(\arcsin x)^2 & dv=dx \\ du=2\arcsin x\cdot\frac{dx}{\sqrt{1-x^2}} & v=x \end{bmatrix} = x(\arcsin x)^2 - 2\int \arcsin x \frac{dx}{\sqrt{1-x^2}}$

$= \begin{bmatrix} u=\arcsin x & dv=\frac{x\,dx}{\sqrt{1-x^2}} \\ du=\frac{dx}{\sqrt{1-x^2}} & v=-\sqrt{1-x^2} \end{bmatrix} = x(\arcsin x)^2 - 2(-\sqrt{1-x^2}\cdot\arcsin x + \int dx)$

$= x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C$

Зад. 6. $\int \frac{x\cdot e^{\operatorname{arctg} x}}{(1+x^2)^{\frac{3}{2}}}dx \iff *$

Решење: $* = \begin{bmatrix} \text{смена: } \operatorname{arctg} x = t,\ x=\operatorname{tg} t \\ \frac{dx}{1+x^2}=dt \end{bmatrix} = \int \frac{\operatorname{tg} t\cdot e^t}{\sqrt{1+\operatorname{tg}^2 t}} = \int \frac{\frac{\sin t}{\cos t}e^t}{\frac{1}{\cos t}}dt = \int e^t \sin t\,dt.$

два пута п.и. $I = \frac{1}{2}e^t(\sin t - \cos t) + C = \frac{1}{2}e^{\operatorname{arctg} x}\left(\frac{x-1}{\sqrt{1+x^2}}\right) + C$

вежба бр. 17 26.04.2007.

Интеграција рационалних функција

$\int \frac{P_n(x)}{Q_m(x)}$, $m > n$ $\quad \int \frac{P_n(x)}{(x-\alpha)^p(x^2+ax+b)^\ell} = \int \frac{A}{x-\alpha} + \frac{B}{(x-\alpha)^2} + \dots + \frac{C}{(x-\alpha)^p} + \frac{A_1x+B_1}{x^2+ax+b} + \dots + \frac{E_1x+F_1}{(x^2+ax+b)^\ell}$

Зад. 1. $\int \frac{x^3}{x^2+1}dx$

Решење: $x^3 : (x^2+1) = x$, остатак $-x$

$$I = \int \left(x - \frac{x}{x^2+1}\right)dx = \int x\,dx - \int \frac{x\,dx}{x^2+1} = \frac{x^2}{2} - \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} = \frac{x^2}{2} - \frac{1}{2}\ln(1+x^2) + C$$

Зад. 2. $\int \frac{dx}{x^3+1}$

Решење: $I = \int \frac{dx}{(x+1)(x^2-x+1)} = \int \left(\frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}\right)dx$

$\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \ /\cdot (x+1)(x^2-x+1)$

$1 = A(x^2-x+1) + (Bx+C)(x+1)$

$1 = x^2(A+B) + x(-A+B+C) + (A+C) \iff$ $A+B=0 \Rightarrow B=-A$; $-A+B+C=0$; $A+C=1 \Rightarrow C=1-A$ $\qquad A=\frac{1}{3}$, $B=-\frac{1}{3}$, $C=\frac{2}{3}$

$I = \int \frac{\frac{1}{3}}{x+1}dx + \int \frac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1}dx = \frac{1}{3}\int\frac{dx}{x+1} - \frac{1}{3}\int\frac{x-2}{x^2-x+1}dx = \frac{1}{3}\int\frac{d(x+1)}{x+1} - \frac{1}{3}I_1$

$I_1 = \int \frac{x-2}{x^2-x+1}dx = \int \frac{\frac{1}{2}(2x-1)+\frac{1}{2}-2}{x^2-x+1}dx = \frac{1}{2}\int\frac{d(x^2-x+1)}{x^2-x+1} - \frac{3}{2}\int\frac{dx}{x^2-x+1}$

$= \frac{1}{2}\ln(x^2-x+1) - \frac{3}{2}\int\frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}} = \frac{1}{2}\ln(x^2-x+1) - \frac{3}{2}\cdot\frac{1}{\sqrt{\frac{3}{4}}}\operatorname{arctg}\frac{x-\frac{1}{2}}{\sqrt{\frac{3}{4}}} + C$

$I = \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}} + C$

Зад. 3. $I = \int \frac{dx}{x^5 - x^2}$

Решење: $I = \int \frac{dx}{x^2(x-1)(x^2+x+1)} = \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{x^2+x+1} \right) dx$

$$\frac{1}{x^2(x-1)(x^2+x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{x^2+x+1}$$

$$1 = Ax(x-1)(x^2+x+1) + B(x^3-1) + Cx^2(x^2+x+1) + (Dx+E)x^2(x-1)$$

$$= \dots = x^4\underbrace{(A+C+D)}_{=0} + x^3\underbrace{(B+C-D+E)}_{=0} + x^2\underbrace{(C-E)}_{=0} - \underbrace{A}_{=0}x - \underbrace{B}_{=1}$$

$$C = \frac{1}{3} \quad D = -\frac{1}{3} \quad E = \frac{1}{3}$$

* постоји и други начин:
посматрамо линеарне чланове и узимамо њихове нуле $(0, 1)$ и убацујемо у једначину добијајући једноставнији систем

$$I = -\int \frac{dx}{x^2} + \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{x-1}{x^2+x+1}dx = \frac{1}{x} + \frac{1}{3}\ln|x-1| - \frac{1}{3}I_1$$

$$= \dots = \frac{1}{x} + \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + C$$

Зад. 4. $\int \frac{3-x}{x^6 - x^5 + x^4 - x^3} dx$

Решење: $I = \int \frac{3-x}{x^3(x-1)(x^2+1)} dx = \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{Ex+F}{x^2+1} \right) dx$

$$3-x = Ax^2(x-1)(x^2+1) + Bx(x-1)(x^2+1) + C(x-1)(x^2+1) + Dx^3(x^2+1) + (Ex+F)x^3(x-1)$$

$$3-x = x^5(A+D+E) + x^4(-A+B-E+F) + x^3(A-B+C+D-F) + x^2(-A+B-C) + x(-B+C) - C$$

$A+D+E=0$	$A=1$
$-A+B-E+F=0$	$B=-2$
$A-B+C+D-F=0$	$C=-3$
$-A+B-C=0$	$D=1$
$-B+C=-1$	$E=-2$
$-C=3$	$F=1$

$$I = 2\int \frac{dx}{x-1} - \int \frac{dx}{(x-1)^2} + \int \frac{-2x-1}{x^2+1}dx + \int \frac{-2x+4}{(x^2+1)^2}dx$$

$$= 2\ln|x-1| + \frac{1}{x-1} - \ln(x^2+1) - \operatorname{arctg} x - \int \frac{2x}{(x^2+1)^2}dx + 4\int \frac{dx}{(x^2+1)^2}$$

(*) $I_1 = \int \frac{dx}{(x^2+1)^2} = \int \frac{(1+x^2)-x^2}{(x^2+1)^2}dx = \int \frac{dx}{x^2+1} - \int \frac{x^2 dx}{(x^2+1)^2} = \begin{bmatrix} u = x & dv = \frac{x\,dx}{(x^2+1)^2} \\ du = dx & v = \frac{-1}{2(x^2+1)} \end{bmatrix}$

$$= \operatorname{arctg} x - \left(-\frac{x}{2(x^2+1)} + \frac{1}{2}\int \frac{dx}{x^2+1} \right)$$

$$I = 2\ln|x-1| + \frac{1}{x-1} - \ln(x^2+1) - \operatorname{arctg} x + \frac{1}{x^2+1} + 2\operatorname{arctg} x + \frac{2x}{x^2+1} + C$$

$$I = 2\ln|x-1| + \frac{1}{x-1} - \ln(x^2+1) + \operatorname{arctg} x + \frac{2x+1}{x^2+1} + C$$

Интеграција тригонометријских функција

$\int R(\sin x, \cos x)\,dx$

1° Универзална смена $\quad tg\frac{x}{2} = t \quad \left[\sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2},\ dx = \frac{2dt}{1+t^2}\right]$

2° $R(-\sin x, \cos x) \equiv -R(\sin x, \cos x) \quad \cos x = t \quad \left[\sin x = \sqrt{1-t^2},\ dx = -\frac{dt}{\sqrt{1-t^2}}\right]$

3° $R(\sin x, -\cos x) \equiv -R(\sin x, \cos x) \quad \sin x = t \quad \left[\cos x = \sqrt{1-t^2},\ dx = \frac{dt}{\sqrt{1-t^2}}\right]$

4° $R(-\sin x, -\cos x) \equiv R(\sin x, \cos x) \quad tg\,x = t \quad \left[\sin x = \frac{t}{\sqrt{1+t^2}},\ \cos x = \frac{1}{\sqrt{1+t^2}},\ dx = \frac{dt}{1+t^2}\right]$

Увек прво пробамо 2°, 3° и 4°, па ако не може онда 1°

Зад. 1. $\int \frac{6+7\sin x}{\sin x(5+4\cos x)}dx$

Решење: $I = \left[\begin{matrix}\text{смена: } tg\frac{x}{2}=t \\ \sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2},\ dx = \frac{2dt}{1+t^2}\end{matrix}\right] = \int \frac{6+7\cdot\frac{2t}{1+t^2}}{\frac{2t}{1+t^2}\left(5+4\frac{1-t^2}{1+t^2}\right)}\cdot\frac{2dt}{1+t^2} = \cdots = \int\frac{6t^2+14t+6}{t(t^2+9)}dt$

$= \int\left(\frac{A}{t}+\frac{Bt+C}{t^2+9}\right)dt = \cdots = \frac{2}{3}\int\frac{dt}{t} + \frac{16}{3}\int\frac{tdt}{t^2+9} + 14\int\frac{dt}{t^2+9}$

$= \frac{2}{3}\ln|t| + \frac{8}{3}\ln|t^2+9| + \frac{14}{3}\,arctg\frac{t}{3} + C = \frac{2}{3}\ln\left|tg\frac{x}{2}\right| + \frac{8}{3}\ln\left(tg^2\frac{x}{2}+9\right) + \frac{14}{3}\,arctg\frac{tg\frac{x}{2}}{3} + C$

Зад. 2. $\int\frac{dx}{\sin^2 x - 5\sin x + 6}$

Решење: $I = \left[\begin{matrix}tg\frac{x}{2}=t \\ \sin x = \frac{2t}{1+t^2} \quad dx = \frac{2dt}{1+t^2}\end{matrix}\right] = \cdots = 2\int\frac{1+t^2}{4t^2-10t(1+t^2)+6(1+t^2)^2}dt$ компликовано

II начин: $I = \int\frac{dx}{(\sin x-2)(\sin x-3)} = \int\frac{dx}{\sin x-3} - \int\frac{dx}{\sin x-2} = I_1 + I_2$

$I_1 = \int\frac{dx}{\sin x-3} = \left[\begin{matrix}\text{смена: } tg\frac{x}{2}=t \\ \sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2},\ dx = \frac{2dt}{1+t^2}\end{matrix}\right] = \cdots = 2\int\frac{dt}{-3t^2+2t-3}$

$= -\frac{2}{3}\int\frac{dt}{(t-\frac{1}{3})^2+\frac{8}{9}} = -\frac{2}{3}\cdot\frac{1}{\frac{2\sqrt{2}}{3}}\,arctg\frac{t-\frac{1}{3}}{\frac{2\sqrt{2}}{3}} + C$

$I_2 = \int\frac{dx}{\sin x-2} = \cdots = -\int\frac{dt}{t^2-t+1} = -\int\frac{dt}{(t-\frac{1}{2})^2+\frac{3}{4}} = -\frac{1}{\frac{\sqrt{3}}{2}}\,arctg\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C$

$I = -\frac{1}{\sqrt{2}}\,arctg\frac{3tg\frac{x}{2}-1}{2\sqrt{2}} + \frac{2}{\sqrt{3}}\,arctg\frac{2tg\frac{x}{2}-1}{\sqrt{3}} + C$

Зад. 3. $\int\frac{\sin^5 x}{\cos^4 x}dx$

Решење: $R(-\sin x, \cos x) \equiv -R(\sin x, \cos x)$

$I = \int\frac{\sin^4 x\cdot\sin x}{\cos^4 x}dx = \int\frac{(1-\cos^2 x)^2}{\cos^4 x}\sin x\,dx = \left[\begin{matrix}\text{смена: } \cos x = t \\ -\sin x dx = dt\end{matrix}\right]$

$= -\int\frac{(1-t^2)^2}{t^4}dt = -\int\frac{1-2t^2+t^4}{t^4}dt = -\int\frac{dt}{t^4} + 2\int\frac{dt}{t^2} - \int dt$

$= \frac{1}{3t^3} - \frac{2}{t} - t + C = \frac{1}{3\cos^3 x} - \frac{2}{\cos x} - \cos x + C$

Зад. 4. $\int \frac{dx}{\sin x(2\cos^2 x-1)}$

Решење: $I = \left[\begin{array}{l}\text{смена}: \cos x = t\\ -\sin x\, dx = dt\end{array}\right] = -\int \frac{dt}{(1-t^2)(2t^2-1)} = \dots = \int \frac{dt}{t^2-1} - 2\int \frac{dt}{2t^2-1}$

$= \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| - \sqrt{2}\int \frac{d(\sqrt{2}t)}{(\sqrt{2}t)^2-1} = \frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right| - \frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}t-1}{\sqrt{2}t+1}\right| + C$

$= \frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right| - \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}\cos x-1}{\sqrt{2}\cos x+1}\right| + C$

Зад. 5. $\int \frac{2\sin x-\cos x}{3\sin^2 x+4\cos^2 x}dx$

Решење: $I = 2\int \frac{\sin x\, dx}{3\sin^2 x+4\cos^2 x} - \int \frac{\cos x\cdot dx}{3\sin^2 x+4\cos^2 x} = 2I_1 - I_2$

$I_1 = \int \frac{\sin x\, dx}{\underset{3-3\cos^2 x}{\underbrace{3\sin^2 x}}+4\cos^2 x} = \left[\begin{array}{l}\text{смена}: \cos x = t\\ -\sin x\, dx = dt\end{array}\right] = -\int \frac{dt}{3+t^2} = -\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{t}{\sqrt{3}} + C$

$I_1 = -\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{\cos x}{\sqrt{3}} + C$

$I_2 = \int \frac{\cos x\, dx}{3\sin^2 x+4\cos^2 x} = \left[\begin{array}{l}\text{смена}: \sin x = t\\ \cos x\, dx = dt\end{array}\right] = \int \frac{dt}{4-t^2} = \frac{1}{2}\cdot\frac{1}{2}\cdot\ln\left|\frac{t+2}{t-2}\right| + C$

$I_2 = \frac{1}{4}\ln\left|\frac{\sin x+2}{\sin x-2}\right|$

$I = -\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{\cos x}{\sqrt{3}} - \frac{1}{4}\ln\left|\frac{\sin x+2}{\sin x-2}\right| + C$

Зад. 6 $\int \frac{dx}{4\sin^2 x+\cos^2 x}$

Решење: $I = \left[\begin{array}{l}\text{смена}: \operatorname{tg} x = t\\ \frac{dx}{\cos^2 x} = dt\end{array}\right] = \int \frac{dt}{4t^2+1} = \frac{1}{2}\int \frac{d(2t)}{(2t)^2+1} = \frac{1}{2}\operatorname{arctg} 2t + C = \frac{1}{2}\operatorname{arctg}(2\operatorname{tg} x) + C$

Зад. 7. $\int \frac{\sin^2 x \;\; /:\cos^4 x}{(\sin x+\cos x)^4 \;\; /:\cos^4 x}dx$

Решење: $I = \int \frac{\frac{\sin^2 x}{\cos^2 x}\cdot\frac{dx}{\cos^2 x}}{\left(\frac{\sin x}{\cos x}+1\right)^4} = \int \frac{\operatorname{tg}^2 x\cdot\frac{dx}{\cos^2 x}}{(\operatorname{tg} x+1)^4} = \int \frac{t^2 dt}{(t+1)^4} = \left[\begin{array}{l}\text{смена}: t+1 = z\\ dt = dz\end{array}\right]$

$= \int \frac{(z-1)^2}{z^4}dz = \int \frac{z^2-2z+1}{z^4}dz = \int \frac{dz}{z^2} - 2\int \frac{dz}{z^3} + \int \frac{dz}{z^4}$

$= -\frac{1}{z} + \frac{1}{z^2} - \frac{1}{3z^3} + C = -\frac{1}{(\operatorname{tg} x+1)} + \frac{1}{(\operatorname{tg} x+1)^2} - \frac{1}{3(\operatorname{tg} x+1)^3} + C$

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Интеграција ирационалних функција

1° Интеграли типа $\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_1}{q_1}}, \dots, \left(\frac{ax+b}{cx+d}\right)^{\frac{p_r}{q_r}}\right)dx$ своде се на интеграл рационалне ф-је сменом $\frac{ax+b}{cx+d} = t^m$, где је $m = NZS(q_1, \dots, q_r)$

Зад. 1. $\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx$

Решење: $I = \begin{bmatrix} \text{смена: } x = t^6 \\ dx = 6t^5dt \end{bmatrix} = \int \frac{t^6+t^4+t}{t^6(1+t^2)}dt \cdot 6t^5 = 6\int \frac{t^5+t^3+1}{t^2+1}dt$

$= 6\int \frac{t^3(t^2+1)+1}{t^2+1}dt = 6\left(\int t^3dt + \int \frac{dt}{t^2+1}\right) = 6\left(\frac{t^4}{4} + \operatorname{arctg} t\right) + C$

$= 6\left(\frac{\sqrt[3]{x^2}}{4} + \operatorname{arctg}\sqrt[6]{x}\right) + C$

Зад. 2. $I = \int \sqrt[3]{\frac{(x+1)^2}{(x-1)^5}}dx$

Решење:

$I = \int \sqrt[3]{\left(\frac{x+1}{x-1}\right)^2}\frac{dx}{x-1} = \begin{bmatrix} \text{смена: } \frac{x+1}{x-1} = t^3 & dx = \frac{3t^2(t^3-1)-(t^3+1)3t^2}{(t^3-1)^2}dt = \frac{-6t^2dx}{(t^3-1)^2} \\ x+1 = xt^3 - t^3 & x-1 = \frac{2}{t^3-1} \\ x = \frac{t^3+1}{t^3-1} & \end{bmatrix}$

$I = \int t^2 \cdot \frac{-6t^2dt}{(t^3-1)^2}\cdot\frac{t^3-1}{2} = -3\int\frac{t^4dt}{t^3-1} = -3\int\frac{t^4dt}{t^3-1} = -3\int\left(t+\frac{t}{t^3-1}\right)dt$

$= -\frac{3t^2}{2} - 3\int\frac{tdt}{t^3-1} = -\frac{3t^2}{2} - 3I_1$

$I_1 = \int\frac{tdt}{t^3-1} = \frac{1}{3}\int\frac{dt}{t-1} + \frac{1}{3}\int\frac{-t+1}{t^2+t+1}dt = \frac{1}{3}\ln(t-1) - \frac{1}{6}\ln(t^2+t+1) + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2t+1}{\sqrt{3}} + C$

$I = -\frac{3}{2}\sqrt[3]{\left(\frac{x+1}{x-1}\right)^2} - \ln\left(\sqrt[3]{\frac{x+1}{x-1}} - 1\right) + \frac{1}{2}\ln\left(\sqrt[3]{\left(\frac{x+1}{x-1}\right)^2} + \sqrt[3]{\frac{x+1}{x-1}} + 1\right) - \sqrt{3}\operatorname{arctg}\frac{2\sqrt[3]{\frac{x+1}{x-1}}+1}{\sqrt{3}} + C$

2° $\int R(x, \sqrt{a^2-x^2})dx$ смена: $x = a\sin t$, $x = a\cos t$

$\int R(x, \sqrt{x^2-a^2})dx$ смена: $x = \frac{a}{\sin t}$, $x = \frac{a}{\cos t}$

$\int R(x, \sqrt{x^2+a^2})dx$ смена: $x = a\operatorname{tg} t$

Зад. 3. $\int\sqrt{4-x^2}\,dx$

Решење: $I = \begin{bmatrix} \text{смена: } x = 2\sin t \Rightarrow \sin t = \frac{x}{2} \\ dx = 2\cos t\,dt \end{bmatrix} = \int\sqrt{4-4\sin^2 t}\cdot 2\cos t\,dt = 4\int\cos^2 t\,dt = 4\int\frac{1+\cos 2t}{2}dt$

$= 2\int(1+\cos 2t)dt = 2\left(t + \frac{1}{2}\sin 2t\right) + C = 2\left(\arcsin\frac{x}{2} + \frac{x}{2}\cdot\frac{1}{2}\cdot\sqrt{4-x^2}\right) + C$

Зад. 4. $\int\frac{dx}{x\sqrt{x^2-a^2}}$

Решење: $I = \begin{bmatrix} \text{смена: } x = \frac{a}{\sin t} \\ dx = -\frac{a}{\sin^2 t}\cos t\,dt \end{bmatrix} = \int\frac{-\frac{a}{\sin^2 t}\cdot\cos t\,dt}{\frac{a}{\sin t}\cdot\frac{\sqrt{a^2-a^2\sin^2 t}}{\sin t}} = -\frac{1}{a}\int dt = -\frac{1}{a}t + C$

$I = -\frac{1}{a}\arcsin\frac{a}{x} + C$

3° $\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}$ се израчунава помоћу формуле $= Q_{m-1}(x)\cdot\sqrt{ax^2+bx+c} + \lambda\int\frac{dx}{\sqrt{ax^2+bx+c}}$

Прво диференцирамо обе стране претходне једнакости, а затим добијени израз помножимо са $\sqrt{ax^2+bx+c}$, па добијамо

$$P_n(x) = Q'_{m-1}(x)\cdot(ax^2+bx+c) + \frac{1}{2}\,Q_{m-1}(x)\cdot(2ax+b) + \lambda \qquad \lambda=?\quad Q_{m-1}(x)=?$$

Зад. 5. $\int\frac{x^4dx}{\sqrt{x^2+1}}$

Решење: $I = (Ax^3+Bx^2+Cx+D)\cdot\sqrt{x^2+1} + \lambda\int\frac{dx}{\sqrt{x^2+1}}$ / диф

$$\frac{x^4}{\sqrt{x^2+1}} = (3Ax^2+2Bx+C)\sqrt{x^2+1} + (Ax^3+Bx^2+Cx+D)\frac{x}{\sqrt{x^2+1}} + \frac{\lambda}{\sqrt{x^2+1}} \;\Big/\cdot\sqrt{x^2+1}$$

$$x^4 = (3Ax^2+2Bx+C)(x^2+1) + x(Ax^3+Bx^2+Cx+D) + \lambda$$
$$= 4Ax^4+3Bx^3+(3A+2C)x^2+(2B+D)x+(C+\lambda)$$

$4A=1 \qquad A=\frac{1}{4}$
$3B=0 \qquad B=0$
$2A+A+2C=0 \qquad C=-\frac{3}{8}$
$2B+D=0 \qquad D=0$
$C+\lambda=0 \qquad \lambda=\frac{3}{5}$

$$I=\left(\frac{1}{4}x^3-\frac{3}{8}x\right)\sqrt{x^2+1}+\frac{3}{8}\int\frac{dx}{\sqrt{x^2+1}}$$
$$I=\left(\frac{1}{4}x^3-\frac{3}{8}x\right)\sqrt{x^2+1}+\frac{3}{8}\ln\left(x+\sqrt{x^2+1}\right)+C$$

Зад. 6. $\int(x+2)\sqrt{2+3x-x^2}\,dx$

Решење: $I=\int(x+2)\sqrt{2+3x-x^2}\,dx\cdot\frac{\sqrt{}}{\sqrt{}}=\int\frac{(x+2)(2+3x-x^2)}{\sqrt{2+3x-x^2}}=\int\frac{-x^3+x^2+8x+4}{\sqrt{2+3x-x^2}}dx$

$$=(Ax^2+Bx+C)\sqrt{2+3x-x^2}+\lambda\int\frac{dx}{\sqrt{2+3x-x^2}} \;\Big/\text{диф.}$$

$$\frac{-x^3+x^2+8x+4}{\sqrt{2+3x-x^2}}=(2Ax+B)\sqrt{2+3x-x^2}+(Ax^2+Bx+C)\frac{3-2x}{2\sqrt{2+3x-x^2}}+\frac{\lambda}{\sqrt{2+3x-x^2}}\;\Big/\cdot2\sqrt{2+3x-x^2}$$

$$-2x^3+2x^2+16x+8=-6Ax^3+(15A-4B)x^2+(8A+9B-2C)x+4B+3C+2\lambda$$

$-6A=-2 \qquad A=\frac{1}{3}$
$15A-4B=2 \qquad B=\frac{3}{4}$
$8A+9B-2C=16 \qquad C=-\frac{79}{24}$
$4B+3C+2\lambda=8 \qquad \lambda=\frac{119}{16}$

$$I=\left(\frac{1}{3}x^2+\frac{3}{4}x-\frac{79}{24}\right)\sqrt{2+3x-x^2}+\frac{119}{16}\arcsin\frac{2x-3}{\sqrt{17}}+C$$

Зад. 7. $I=\int\frac{dx}{(x+1)^3\sqrt{x^2+2x-3}}$

Решење: $I=\left[\begin{matrix}\text{смена: } x+1=\frac{1}{t},\ x=\frac{1-t}{t}\\ dx=-\frac{dt}{t^2}\end{matrix}\right]=\dots=\int-\frac{t^2}{\sqrt{1-4t^2}}dt$

$$=(At+B)\sqrt{1-4t^2}+\lambda\int\frac{dt}{\sqrt{1-4t^2}}=\dots=\frac{1}{8}t\sqrt{1-4t^2}-\frac{1}{8}\int\frac{dt}{\sqrt{1-4t^2}}$$

$$=\frac{1}{8}t\sqrt{1-4t^2}+\frac{1}{16}\arccos 2t+C=\frac{1}{8}\frac{\sqrt{x^2+2x-3}}{(x+1)^2}+\frac{1}{16}\arccos\frac{2}{x+1}+C$$

вежба бр. 19 03. 05. 2007.

Рекурентне формуле

Зад. 1. Извести рекурентну формулу за $I_m = \int \sin^n x\, dx$, па на основу тога израчунати I_4.

Решење: Код оваквих врста задатака обично се користи парцијална интеграција

$$I_m = \int \sin^m x = \begin{bmatrix} u=\sin^{m-1}x & dv=\sin x\, dx \\ du=(m-1)\sin^{m-2}x\cdot\cos x\, dx & v=-\cos x \end{bmatrix} = -\cos x\cdot \sin^{m-1}x + (m-1)\int \sin^{m-2}x\cdot\cos^2 x\, dx$$

$$= -\cos x\cdot\sin^{m-1}x + (m-1)\int \sin^{m-2}x\,(1-\sin^2 x)\, dx$$

$$= -\cos x\cdot\sin^{m-1}x + (m-1)\int \sin^{m-2}x\, dx - (m-1)\int \sin^m x\, dx$$

$$I_m = -\cos x\cdot\sin^{m-1}x + (m-1)\, I_{m-2} - (m-1)\, I_m$$

$$m\cdot I_m = -\cos x\,\sin^{m-1}x + (m-1)\, I_{m-2}$$

$$I_m = -\frac{1}{m}\cos x\cdot\sin^{m-1}x + \frac{m-1}{m}\, I_{m-2}\quad ,\quad m\geq 2$$

$$I_4 = \int \sin^4 x\, dx = -\frac{1}{4}\cos x\sin^3 x + \frac{3}{4} I_2 = -\frac{1}{4}\cos x\cdot\sin^3 x + \frac{3}{4}\left(-\frac{1}{2}\cos x\cdot\sin x + \frac{1}{2} I_0\right)$$

$$I_0 = \int dx = x + C$$

$$I_4 = -\frac{1}{4}\cos x\cdot\sin^3 x - \frac{3}{8}\sin x\cdot\cos x + \frac{3}{8}x + C$$

Зад. 2. $I_m = \int \frac{dx}{\cos^m x}\quad ,\quad I_5 = ?$

Решење: $$I_m = \int\frac{dx}{\cos^m x} = \int\frac{\cos^2 x + \sin^2 x}{\cos^m x}\, dx = \int\frac{\sin^2 x}{\cos^m x}\, dx + \int\frac{dx}{\cos^{m-2}x} = \begin{bmatrix} u=\sin x & dv=\frac{\sin x\, dx}{\cos^m x} \\ du=\cos x\, dx & v=\frac{1}{m-1}\cdot\cos^{m-1}x \end{bmatrix}$$

$$= \frac{1}{m-1}\cdot\frac{\sin x}{\cos^{m-1}x} - \frac{1}{m-1}\int\frac{dx}{\cos^{m-2}x} + I_{m-2}$$

$$I_m = \frac{1}{m-1}\cdot\frac{\sin x}{\cos^{m-1}x} + \frac{m-2}{m-1}\, I_{m-2}\quad ,\quad m\geq 2$$

$$I_5 = \int\frac{dx}{\cos^5 x} = \frac{1}{4}\cdot\frac{\sin x}{\cos^4 x} + \frac{3}{4} I_3 = \frac{1}{4}\cdot\frac{\sin x}{\cos^4 x} + \frac{3}{4}\left(\frac{1}{2}\,\frac{\sin x}{\cos^2 x} + \frac{1}{2} I_1\right)$$

$$I_1 = \int\frac{dx}{\cos x} = \int\frac{\cos x}{\cos^2 x}\, dx = \int\frac{\cos x}{1-\sin^2 x}\, dx = \int\frac{dt}{1-t^2} = \frac{1}{2}\ln\left|\frac{1-t}{1+t}\right|$$

$$I_5 = \frac{1}{4}\cdot\frac{\sin x}{\cos^4 x} + \frac{3}{8}\cdot\frac{\sin x}{\cos^2 x} + \frac{3}{16}\ln\left|\frac{1-\sin x}{1+\sin x}\right| + C$$

Интеграција хиперболичких функција

$$\int R(\operatorname{sh}x, \operatorname{ch}x)\, dx$$

1° универзална смена: $\operatorname{th}\frac{x}{2} = t\quad \left[\operatorname{sh}x = \frac{2t}{1-t^2},\ \operatorname{ch}x = \frac{1+t^2}{1-t^2},\ dx = \frac{2dt}{1-t^2}\right]$

2° смена: $\operatorname{th}x = t\quad \left[\operatorname{sh}x = \frac{t}{\sqrt{1-t^2}}\ ,\ \operatorname{ch}x = \frac{1}{\sqrt{1-t^2}}\ ,\ dx = \frac{dt}{\sqrt{1-t^2}}\right]$

везе: $\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1$
$\operatorname{sh}2x = 2\operatorname{sh}x\cdot\operatorname{ch}x$
$1+\operatorname{ch}2x = 2\operatorname{ch}^2 x$
$\operatorname{ch}2x - 1 = 2\operatorname{sh}^2 x$

Зад. 1. $\int \frac{dx}{shx\cdot chx}$

Решење: $I = \int \frac{chx\cdot dx}{shx\cdot ch^2x} = \int \frac{dx}{thx\cdot ch^2x} = \left[\text{смена } \begin{matrix} thx = t \\ \frac{dx}{\cos^2 x} = dt \end{matrix}\right] = \int \frac{dt}{t} = \ln|t| + C = \ln|thx| + C$

Зад. 2. $\int \frac{dx}{5chx+3shx+4}$

Решење: $I = \left[\begin{matrix}\text{смена: } th\frac{x}{2} = t \\ \ldots\ldots \end{matrix}\right] = \ldots = 2\int \frac{dt}{t^2+6t+9} = 2\int \frac{dt}{(t+3)^2} = -\frac{2}{t+3} + C = -\frac{2}{th\frac{x}{2}+3} + C$

Одређени интеграл

1° $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

2° $\int_a^b c\cdot f(x)dx = c\cdot \int_a^b f(x)dx \quad , \ c\in R$

3° $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

4° $\int_a^a f(x)dx = 0$

5° $\int_a^b f(x) = -\int_b^a f(x)dx$

6° $f(x)\geq 0 \ , \ x\in[a,b] \Rightarrow \int_a^b f(x)dx \geq 0$

7° $f(x)\geq g(x) \ , \ x\in[a,b] \Rightarrow \int_a^b f(x)dx \geq \int_a^b g(x)dx$

8° $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

9° а) $f(x)$ парна ф-ја $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

б) $f(x)$ непарна ф-ја $\int_{-a}^{a} f(x)dx = 0$

$$P = \int_a^b f(x)dx$$

Њутн - Лајбницова формула

Нека је функција $f(x)$ непрекидна на интервалу $[a,b]$ и нека је $F(x) = \int f(x)dx$ примитивна функција. Тада је $\int_a^b f(x)dx = F(x)\big|_a^b = F(b) - F(a)$

Зад. 1. $\int_0^1 \frac{x+1}{x^2+1}dx$

Решење: $I = \int_0^1 \frac{x\,dx}{x^2+1} + \int_0^1 \frac{dx}{x^2+1} = \frac{1}{2}\ln(x^2+1)\big|_0^1 + \operatorname{arctg} x\big|_0^1$

$= \frac{1}{2}(\ln 2 - \cancel{\ln 1}^{\,0}) + \operatorname{arctg} 1 - \cancel{\operatorname{arctg} 0}^{\,0} = \frac{1}{2}\ln 2 + \frac{\pi}{4}$

Зад. 2. $\int\limits_0^{2\pi}\sqrt{1-\cos 2x}\,dx$

Решење: $I=\int\limits_0^{2\pi}\sqrt{2\sin^2 x}$ = (код о.м. водимо рачуна о знаку) $=\sqrt{2}\int\limits_0^{2\pi}|\sin x|\,dx$

$$=\sqrt{2}\left(\int\limits_0^{\pi}|\sin x|\,dx+\int\limits_{\pi}^{2\pi}|\sin x|\,dx\right)=\sqrt{2}\left(\int\limits_0^{\pi}\sin x\,dx-\int\limits_{\pi}^{2\pi}\sin x\,dx\right)$$

$$=\sqrt{2}\left(-\cos x\Big|_0^{\pi}+\cos x\Big|_{\pi}^{2\pi}\right)=\sqrt{2}\left(-(\cos\pi-\cos 0)+(\cos 2\pi-\cos\pi)\right)$$

$$=\sqrt{2}\left(-(-1-1)+(1-(-1))\right)=\sqrt{2}\,(2+2)=4\sqrt{2}$$

Смена променљиве код одређених интеграла

Зад. 1. $\int\limits_1^{e^3}\frac{dx}{x\sqrt{1+\ln x}}$

Решење: $I=\begin{bmatrix}\text{смена} & \\ \ln x=t & t=\ln 1=0\\ \frac{dx}{x}=dt & t=\ln e^3=3\end{bmatrix}=\int\limits_0^3\frac{dt}{\sqrt{1+t}}=\int\limits_0^3(1+t)^{-\frac12}\cdot d(1+t)=\frac{(1+t)^{\frac12}}{\frac12}\Big|_0^3=2(\sqrt{4}-\sqrt{1})=2\cdot 1=2$

Зад. 2. $\int\limits_0^{1/2}\sqrt{1-x^2}\,dx$

Решење: $I=\begin{bmatrix}\text{смена: } x=\sin t & x=0:\ t=0\\ dx=\cos t\,dt & x=\frac12:\ t=\frac{\pi}{6}\end{bmatrix}=\int\limits_0^{\pi/6}\sqrt{1-\sin^2 t}\cdot\cos t\,dt=\int\limits_0^{\pi/6}|\cos t|\cdot\cos t\,dt$

$$=\int\limits_0^{\pi/6}\cos^2 t\,dt=\int\limits_0^{\pi/6}\frac{1+\cos 2t}{2}\,dt=\frac12\left(t+\frac{\sin 2t}{2}\right)\Big|_0^{\frac{\pi}{6}}=\frac12\left(\frac{\pi}{6}+\frac{\sin\frac{\pi}{3}}{2}-\left(0+\frac{\sin 0}{2}\right)\right)=\frac12\left(\frac{\pi}{6}+\frac{\sqrt{3}}{4}\right)$$

Зад. 3. $\int\limits_0^{\ln 2}\frac{e^x-1}{\sqrt{e^x+1}}\,dx$

Решење: $I=\begin{bmatrix}\text{смена: } e^x+1=t^2\Rightarrow e^x=t^2-1 & \\ e^x dx=2t\,dt & x=0\Rightarrow t=\sqrt{2}\\ dx=\frac{2t}{t^2-1}\,dt & x=\ln 2\Rightarrow t=\sqrt{3}\end{bmatrix}=\int\limits_{\sqrt2}^{\sqrt3}\frac{(t^2-1)-1}{\cancel{t}}\cdot\frac{2\cancel{t}\,dt}{t^2-1}=2\int\limits_{\sqrt2}^{\sqrt3}\frac{t^2-2}{t^2-1}\,dt$

$$=2\left(\int\limits_{\sqrt2}^{\sqrt3}\frac{t^2-1}{t^2-1}\,dt-\int\limits_{\sqrt2}^{\sqrt3}\frac{dt}{t^2-1}\right)=2\left(t-\frac12\ln\frac{t-1}{t+1}\right)\Big|_{\sqrt2}^{\sqrt3}$$

$$=2\left(\sqrt3-\frac12\ln\frac{\sqrt3-1}{\sqrt3+1}-\left(\sqrt2-\frac12\ln\frac{\sqrt2-1}{\sqrt2+1}\right)\right)=\dots=2(\sqrt3-\sqrt2)-\ln(2-\sqrt3)\cdot(3+2\sqrt2)$$

Парцијална интеграција код одређених интеграла

Зад. 1. $\int\limits_1^2 x^2\ln x\,dx$

Решење: $I=\begin{bmatrix}u=\ln x & dv=x^2dx\\ du=\frac{dx}{x} & v=\frac{x^3}{3}\end{bmatrix}=\frac{x^3}{3}\ln x\Big|_1^2-\frac13\int\limits_1^2 x^2dx=\frac83\ln 2-\frac13\ln 1-\frac13\,\frac{x^3}{3}\Big|_1^2$

$$=\frac83\ln 2-\frac13\left(\frac83-\frac13\right)=\frac83\ln 2-\frac79$$

Зад. 2. $\int\limits_0^1\frac{x\cdot\ln\left(x+\sqrt{1+x^2}\right)}{(1+x^2)^2}\,dx$

Решење: $I=\begin{bmatrix}u=\ln\left(x+\sqrt{1+x^2}\right) & dv=\frac{x\,dx}{(1+x^2)^2}=\frac12\,\frac{d(1+x^2)}{(1+x^2)^2}\\ du=\frac{dx}{\sqrt{1+x^2}} & v=-\frac12\cdot\frac{1}{1+x^2}\end{bmatrix}=-\frac12\,\frac{\ln\left(x+\sqrt{1+x^2}\right)}{1+x^2}\Big|_0^1+\frac12\int\limits_0^1\frac{dx}{(1+x^2)^{\frac32}}=-\frac12\cdot\frac{\ln(1+\sqrt2)}{2}+\frac12 I_1$

$I_1=\int\limits_0^1\frac{dx}{(1+x^2)^{\frac32}}=\begin{bmatrix}\text{смена: } x=tg\,t & x=0\Rightarrow t=0\\ dx=\frac{dt}{\cos^2 t} & x=1\Rightarrow t=\frac{\pi}{4}\end{bmatrix}=\int\limits_0^{\pi/4}\frac{\frac{dt}{\cos^2 t}}{(1+tg^2 t)^{\frac32}}=\dots=\int\limits_0^{\pi/4}\cos t\,dt=\sin t\Big|_0^{\frac{\pi}{4}}=\frac{\sqrt2}{2}$

$$I=-\frac14\ln(1+\sqrt2)+\frac{\sqrt2}{4}$$

Зад. 3. $\int\limits_0^{\frac{\pi}{2}} \sin 2x \cdot \operatorname{arctg}(\cos^2 x)\,dx$

Решење: $I = \begin{bmatrix} \text{смена: } \cos^2 x = t & x=0 \Rightarrow t=1 \\ -2\cos x \cdot \sin x\,dx = dt & x=\frac{\pi}{2} \Rightarrow t=0 \\ -\sin 2x\,dx = dt & \end{bmatrix} = \int\limits_1^0 \operatorname{arctg} t\,(-dt) = -\int\limits_1^0 \operatorname{arctg} t$

$= \int\limits_0^1 \operatorname{arctg} t\,dt = \begin{bmatrix} u = \operatorname{arctg} t & dv = dt \\ du = \frac{dt}{1+t^2} & v = t \end{bmatrix} = t \operatorname{arctg} t \Big|_0^1 - \int \frac{t\,dt}{1+t^2}$

$= 1 \cdot \operatorname{arctg} 1 - 0 - \frac{1}{2}\int\limits_0^1 \frac{d(1+t^2)}{1+t^2} = \frac{\pi}{4} - \frac{1}{2}\ln(1+t^2)\Big|_0^1 = \frac{\pi}{4} - \frac{1}{2}(\ln 2 - \ln 1) = \frac{\pi}{4} - \frac{1}{2}\ln 2$

Зад. 4. $\int\limits_{\frac{1}{e}}^{e} |\ln x|\,dx$

Решење: $I = -\int\limits_{\frac{1}{e}}^{1} \ln x\,dx + \int\limits_1^e \ln x\,dx$

$\int \ln x\,dx = \begin{bmatrix} u = \ln x & dv = dx \\ du = \frac{dx}{x} & v = x \end{bmatrix} = x\ln x - x + C$

$I = -(x\ln x - x)_{\frac{1}{e}}^{1} + (x\ln x - x)_1^e = -\left(1\cdot\ln 1 - 1 - \left(\frac{1}{e}\ln\frac{1}{e} - \frac{1}{e}\right)\right) + (e\ln e - e - (1\cdot\ln 1 - 1))$

$= -\left(-1 - \left(-\frac{1}{e} - \frac{1}{e}\right)\right) + (e - e + 1) = 1 - \frac{2}{e} + 1 = 2 - \frac{2}{e}$

Зад. 5. $\int\limits_0^3 \arcsin\sqrt{\frac{x}{x+1}}\,dx$

Решење: $I = \begin{bmatrix} u = \arcsin\sqrt{\frac{x}{x+1}} & dv = dx \\ du = \frac{1}{2}\cdot\frac{dx}{\sqrt{x}(1+x)} & v = x \end{bmatrix} = x\cdot\arcsin\sqrt{\frac{x}{x+1}}\Big|_0^3 - \frac{1}{2}\int\limits_0^3 \frac{x\,dx}{\sqrt{x}(1+x)}$

$= 3\arcsin\frac{\sqrt{3}}{2} - 0 - \frac{1}{2}\int\limits_0^3 \frac{\sqrt{x}\,dx}{1+x} = 3\cdot\frac{\pi}{3} - \frac{1}{2}I_1$

$I_1 = \int\limits_0^3 \frac{\sqrt{x}\,dx}{1+x} = \begin{bmatrix} \text{смена: } x = t^2 & x=0 \Rightarrow t=0 \\ dx = 2t\,dt & x=3 \Rightarrow t=\sqrt{3} \end{bmatrix} = 2\int\limits_0^{\sqrt{3}} \frac{t\cdot t\,dt}{1+t^2} = 2\int\limits_0^{\sqrt{3}} \frac{(t^2+1)-1}{t^2+1}\,dt$

$= 2\left(\int\limits_0^{\sqrt{3}} dt - \int\limits_0^{\sqrt{3}} \frac{dt}{1+t^2}\right) = 2(t - \operatorname{arctg} t)_0^{\sqrt{3}} = 2(\sqrt{3} - \operatorname{arctg}\sqrt{3} - 0) = 2\left(\sqrt{3} - \frac{\pi}{3}\right)$

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Несвојствени интеграли

I) на неограниченим интервалима $[a,+\infty)$, $(-\infty, b]$, $(-\infty,+\infty)$

1а) $\int\limits_a^{+\infty} f(x)\,dx = \lim\limits_{b'\to+\infty} \int\limits_a^{b'} f(x)\,dx$, $b' > a$

1б) $\int\limits_{-\infty}^{b} f(x)\,dx = \lim\limits_{a'\to-\infty} \int\limits_{a'}^{b} f(x)\,dx$, $a' < b$

1в) $\int\limits_{-\infty}^{+\infty} f(x)\,dx = \lim\limits_{a'\to-\infty} \int\limits_{a'}^{c} f(x)\,dx + \lim\limits_{b'\to+\infty} \int\limits_c^{b'} f(x)\,dx$, $a' < c < b'$

Ако поменути лимеси постоје и коначни су онда кажемо да је несвојствени интеграл конвергентан. У супротном кажемо да дивергира.

Зад. 1. $\int\limits_{1}^{\boxed{+\infty}} \frac{dx}{x^4}$

Решење: $I = \lim\limits_{b\to+\infty} \int\limits_1^b \frac{dx}{x^4} = \lim\limits_{b\to+\infty} -\frac{1}{3x^3}\Big|_1^b = -\frac{1}{3}\lim\limits_{b\to+\infty}\left(\frac{1}{b^3} - \frac{1}{1^3}\right) = \frac{1}{3}$

Зад. 2. $\int\limits_{1}^{\boxed{+\infty}} \frac{dx}{x^2(x+1)}$

Решење: $I = \lim\limits_{b\to+\infty} \int\limits_1^b \frac{dx}{x^2(x+1)} = \lim\limits_{b\to+\infty} \int\limits_1^b \left(-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right)dx = \lim\limits_{b\to+\infty}\left(-\ln x - \frac{1}{x} + \ln(x+1)\right)_1^b$

$$= \lim\limits_{b\to+\infty}\left(\ln\frac{x+1}{x} - \frac{1}{x}\right)_1^b = \lim\limits_{b\to+\infty}\left(\ln\frac{b+1}{b} - \frac{1}{b} - \left(\ln\frac{2}{1} - 1\right)\right) = 1 - \ln 2$$

Зад. 3. $\int\limits_{\boxed{-\infty}}^{\boxed{+\infty}} \frac{dx}{x^2+2x+2}$

Решење: $I = \lim\limits_{a\to-\infty} \int\limits_a^0 \frac{dx}{x^2+2x+2} + \lim\limits_{b\to+\infty} \int\limits_0^b \frac{dx}{x^2+2x+2} = \lim\limits_{a\to-\infty}\left(\operatorname{arctg}(x+1)\right)_a^0 + \lim\limits_{b\to+\infty}\left(\operatorname{arctg}(x+1)\right)_0^b$

$$= \lim\limits_{a\to-\infty}\left(\operatorname{arctg}1 - \operatorname{arctg}(a+1)\right) + \lim\limits_{b\to\infty}\left(\operatorname{arctg}(b+1) - \operatorname{arctg}1\right) = \pi$$

II) ограничен интервал интеграције али подинтегрална ф-ја не мора бити ограничена на њему

2а) $\int\limits_a^b f(x)dx = \lim\limits_{b'\to b_-} \int\limits_a^{b'} f(x)dx$

2б) $\int\limits_a^b f(x)dx = \lim\limits_{a'\to a_+} \int\limits_{a'}^{b} f(x)dx$

2в) $\int\limits_a^b f(x)dx = \lim\limits_{a'\to a_+} \int\limits_{a'}^{c} f(x)dx + \lim\limits_{b'\to b_-} \int\limits_{c}^{b'} f(x)dx \quad , a<c<b$

Зад. 1. $\int\limits_{\boxed{0}}^{\frac{1}{e}} \frac{dx}{x\ln^2 x}$

Решење: $I = \lim\limits_{a\to 0+} \int\limits_a^{\frac{1}{e}} \frac{dx}{x\ln^2 x} = \left[\text{смена: } \begin{matrix} \ln x = t & x=a \to t=\ln a \\ \frac{dx}{x} = dt & x=\frac{1}{e} \to t=-1 \end{matrix}\right] = \lim\limits_{a\to 0+} \int\limits_{\ln a}^{-1} \frac{dt}{t^2}$

$$= \lim\limits_{a\to 0+}\left(-\frac{1}{t}\right)_{\ln a}^{-1} = \lim\limits_{a\to 0+}\left(1 + \frac{1}{\ln a}\right) = 1$$

Зад. 2. $\int\limits_{\boxed{1}}^{\boxed{3}} \frac{dx}{\sqrt{4x-x^2-3}}$

Решење: $I = \int \frac{dx}{\sqrt{4x-x^2-3}} = \int \frac{dx}{\sqrt{1-(x-2)^2}} = \int \frac{d(x-2)}{\sqrt{1-(x-2)^2}} = \arcsin(x-2) + C$

$I = \lim\limits_{a\to 1+} \int\limits_a^2 \frac{dx}{\sqrt{4x-x^2-3}} + \lim\limits_{b\to 3-} \int\limits_2^b \frac{dx}{\sqrt{4x-x^2-3}} = \lim\limits_{a\to 1+} \arcsin(x-2)\Big|_a^2 + \lim\limits_{b\to 3-} \arcsin(x-2)\Big|_2^b$

$$= \lim\limits_{a\to 1+}\left(\arcsin 0 - \arcsin(a-2)\right) + \lim\limits_{b\to 3_-}\left(\arcsin(b-2) - \arcsin 0\right) = -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi$$

III) на неограниченим интервалима $(a,+\infty)$, $(-\infty,b)$ при чему подинтегрална ф-ја не мора бити ограничена у некој околини тачке a односно b.

3а) $\displaystyle\int_a^{+\infty} f(x)dx = \lim_{a'\to a+}\int_{a'}^{c} f(x)dx + \lim_{b'\to+\infty}\int_c^{b'} f(x)dx \quad , \quad c\in(a,+\infty)$

3б) $\displaystyle\int_{-\infty}^{b} f(x)dx = \lim_{a'\to-\infty}\int_{a'}^{c} f(x)dx + \lim_{b'\to b-}\int_c^{b'} f(x)dx \quad , \quad c\in(-\infty,b)$

Зад. 1. $\displaystyle\int_1^{+\infty}\frac{dx}{x\sqrt{x-1}}$

Решење: $\displaystyle I = \lim_{a\to1+0}\int_a^2\frac{dx}{x\sqrt{x-1}} + \lim_{b\to+\infty}\int_2^b\frac{dx}{x\sqrt{x-1}}$

$$*\int\frac{dx}{x\sqrt{x-1}} = \begin{bmatrix}\text{смена: } x-1=t^2\\ dx=2t\,dt\end{bmatrix} = \int\frac{2\cancel{t}dt}{(t^2+1)\cancel{t}} = 2\,\mathrm{arctg}\,t + C$$

$$I = \lim_{a\to1+0}\left(2\,\mathrm{arctg}\sqrt{x-1}\right)_a^2 + \lim_{b\to+\infty}\left(2\,\mathrm{arctg}\sqrt{x-1}\right)_2^b$$

$$= \cancel{2\,\mathrm{arctg}\,1} - 2\lim_{a\to1+0}\underbrace{\mathrm{arctg}\sqrt{a-1}}_{0} + 2\lim_{b\to+\infty}\mathrm{arctg}\sqrt{b-1} - \cancel{2\,\mathrm{arctg}\,1} = -2\cdot0+2\cdot\frac{\pi}{2} = \pi$$

Зад. 2. $\displaystyle\int_0^1 \ln\left(1+\frac{1}{\sqrt{x}}\right)dx$

Решење: $\displaystyle I = \lim_{a\to0+}\int_a^1 \ln\left(1+\frac{1}{\sqrt{x}}\right)dx = *$

$$* = \int\ln\left(1+\frac{1}{\sqrt{x}}\right)dx = \begin{bmatrix}u=\ln(1+\frac{1}{\sqrt{x}}) & dv=dx\\ du=-\frac{1}{2}\frac{dx}{x(\sqrt{x}+1)} & v=x\end{bmatrix} = x\ln\left(1+\frac{1}{\sqrt{x}}\right)+\frac{1}{2}\int\frac{dx}{\sqrt{x}+1} = [\text{смена}: x=t^2]$$

$$= \dots = x\ln\left(1+\frac{1}{\sqrt{x}}\right)+\sqrt{x}-\ln(\sqrt{x}+1)+C$$

$$I = \lim_{a\to0+}\left(x\ln\left(1+\frac{1}{\sqrt{x}}\right)+\sqrt{x}-\ln(\sqrt{x}+1)\right)_a^1$$

$$= \lim_{a\to0+}\left((\ln2+1-\ln2)-a\ln\left(1+\frac{1}{\sqrt{a}}\right)+\sqrt{a}-\ln(\sqrt{a}+1)\right)$$

$$= 1-\lim_{a\to0+}a\ln\left(1+\frac{1}{\sqrt{a}}\right) = 1$$

$$\lim_{a\to0+}a\ln\left(1+\frac{1}{\sqrt{a}}\right) = \lim_{a\to0+}\frac{\ln(1+\frac{1}{\sqrt{a}})}{\frac{1}{a}} \overset{\text{л.п.}}{=} \dots = 0$$

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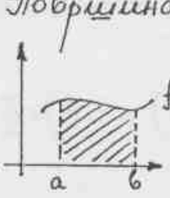
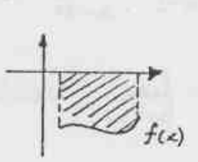
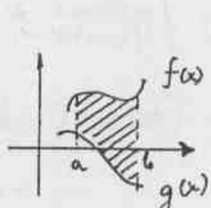
Примена одређеног интеграла

I) Површина равне фигуре

1° 

2° 

3° 

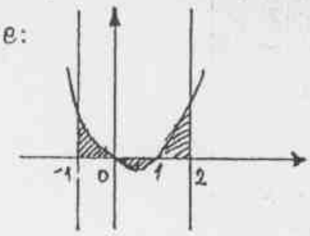
$P=\displaystyle\int_a^b f(x)dx$

$P=-\displaystyle\int_a^b f(x)dx$

$P=\displaystyle\int_a^b (f(x)-g(x))dx$

Зад. 1. Израчунати површину фигуре ограничене кривом $y=x^2-x$, осом Ox и правама $x=-1$ и $x=2$.

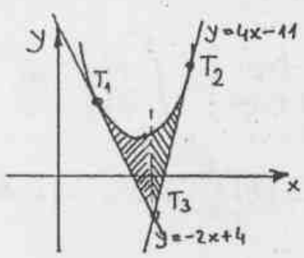
Решење:



$$P=\int_{-1}^{0} y\,dx-\int_{0}^{1} y\,dx+\int_{1}^{2} y\,dx$$
$$=\int_{-1}^{0}(x^2-x)dx-\int_{0}^{1}(x^2-x)dx+\int_{1}^{2}(x^2-x)dx$$
$$=\left(\frac{x^3}{3}-\frac{x^2}{2}\right)\Big|_{-1}^{0}-\left(\frac{x^3}{3}-\frac{x^2}{2}\right)\Big|_{0}^{1}+\left(\frac{x^3}{3}-\frac{x^2}{2}\right)\Big|_{1}^{2}$$
$$=\frac{5}{6}+\frac{1}{6}+\frac{5}{6}=\frac{11}{6}$$

Зад. 2. Израчунати површину фигуре ограничене параболом $y=x^2-4x+5$ и њеним тангентама $y=-2x+4$ и $y=4x-11$

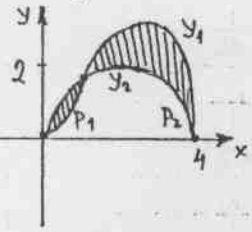
Решење:



$$\left.\begin{array}{l} y=x^2-4x+5 \\ y=-2x+4 \end{array}\right\} \Rightarrow T_1(1,2) \qquad T_2(4,5) \qquad T_3\left(\frac{5}{2},-1\right)$$
$$P=P_1+P_2=\int_{1}^{\frac{5}{2}}(y-y_1)dx+\int_{\frac{5}{2}}^{4}(y-y_2)dx$$
$$P=\int_{1}^{\frac{5}{2}}(x^2-2x+1)dx+\int_{5/2}^{4}(x^2-8x+16)dx=\dots=\frac{9}{4}$$

Зад. 3. Израчунати површину фигуре ограничене графицима функција $y_1=x\sqrt{4x-x^2}$, $y_2=\sqrt{4x-x^2}$.

Решење: $\mathcal{D}_f=[0,4]$ $\qquad O(0,0) \quad A(1,\sqrt{3}) \quad B(4,0)$



$$P=P_1+P_2=\int_{0}^{1}(y_2-y_1)dx+\int_{1}^{4}(y_1-y_2)dx$$
$$=\int_{0}^{1}(1-x)\sqrt{4x-x^2}\,dx+\int_{1}^{4}(x-1)\sqrt{4x-x^2}\,dx$$

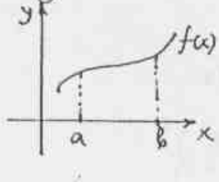
$$\int(x-1)\sqrt{4x-x^2}\,dx=I$$

I начин: $I=\int(x-1)\sqrt{4-(x-2)^2}\,dx=\left[\begin{array}{l}\text{смена: } x-2=2\sin t\\ dx=2\cos t\end{array}\right]=\dots=$
$$=4\int(1+2\sin t)\cos^2 t\,dt=4\int\cos^2 t\,dt+8\int\sin t\cdot\cos^2 t\,dt$$
$$=4\int\frac{1+\cos 2t}{2}dt+8\int\dots=2t+\sin 2t-\frac{8}{3}\cos^3 t+C$$

II начин: $I=\int(x-1)\sqrt{4x-x^2}\,dx\cdot\frac{\sqrt{4x-x^2}}{\sqrt{4x-x^2}}$
$$=\int(x-1)\frac{4x-x^2}{\sqrt{4x-x^2}}dx=(Ax^2+Bx+C)\sqrt{4x-x^2}+\lambda\int\frac{dx}{\sqrt{4x-x^2}} \quad \Big/ \text{диф.}$$
$$\frac{-x^3+5x^2-4x}{\sqrt{4x-x^2}}=(2Ax+B)\sqrt{4x-x^2}+(Ax^2+Bx+C)\frac{4-2x}{2\sqrt{4x-x^2}}+\lambda\frac{1}{\sqrt{4x-x^2}} \quad \Big/\cdot\sqrt{4x-x^2}$$

. . .

II) Дужина лука криве



$$l = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

Зад. 1. Израчунати дужину лука криве $y = \ln\frac{e^x-1}{e^x+1}$, $1 \le x \le 2$

Решење: $y' = \frac{e^x+1}{e^x-1}\cdot\frac{e^x(e^x+1)-e^x(e^x-1)}{(e^x+1)^2} = \frac{2e^x}{e^{2x}-1}$

$$1+y'^2 = 1+\frac{4e^{2x}}{(e^{2x}-1)^2} = \dots = \left(\frac{e^{2x}+1}{e^{2x}-1}\right)^2$$

$$\int_1^2 \sqrt{1+y'^2}\,dx = \int_1^2 \frac{e^{2x}+1}{e^{2x}-1}\,dx = \left[\begin{array}{ll} \text{смена: } e^x = t & x=1: t=e \\ e^x dx = dt & x=2: t=e^2 \\ dx = \frac{dt}{t} & \end{array}\right] = \int_e^{e^2} \frac{t^2+1}{t^2-1}\cdot\frac{dt}{t}$$

$$= \int_e^{e^2}\left(\frac{1}{t-1}+\frac{1}{t+1}-\frac{1}{t}\right)dt = \left(\ln(t-1)+\ln(t+1)-\ln(t)\right)\Big|_e^{e^2} = \dots = -1+\ln(e^2+1)$$

Зад. 2. $y = \sqrt{x^2-32} + 8\ln\left(x+\sqrt{x^2-32}\right)$ $\qquad x \in [6,9]$

Решење: $y' = \frac{x+8}{\sqrt{x^2-32}} \Rightarrow 1+y'^2 = \frac{2(x+4)^2}{x^2-32}$

$$l = \int_6^9 \sqrt{1+y'^2}\,dx = \int_6^9 \frac{\sqrt{2(x+4)^2}}{\sqrt{x^2-32}}dx = \int_6^9 \frac{\sqrt{2}\,(x+4)}{\sqrt{x^2-32}}dx$$

$$\int \frac{x+4}{\sqrt{x^2-32}} = A\sqrt{x^2-32} + \lambda\int\frac{dx}{\sqrt{x^2-32}} \;\Big/\; \text{диф.} \;\dots \Big/ \cdot\sqrt{x^2-32}$$

$$l = \sqrt{2}\left(\sqrt{x^2-32}+4\ln\left(x+\sqrt{x^2-32}\right)\right)\Big|_6^9 = \dots = \sqrt{2}\,(5+4\ln 2)$$

Зад. 3. Израчунати површину и обим фигуре ограничене кривама $y = \frac14 x^2 - \frac12\ln x$, $y=0$, $x=1$, $x=2$.

Решење: $P = \int_1^2\left(\frac14x^2-\frac12\ln x\right)dx = \left[\frac{x^3}{12} - \frac12 x\ln x + \frac12 x\right]_1^2 = \dots = \frac{13}{12} - \ln 2$

$$O = l + y(1) + y(2) + 1$$

$$y' = \frac12\cdot\frac{x^2-1}{x}$$

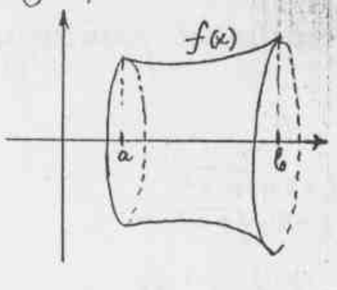
$$1+y'^2 = \dots = \left(\frac{x^2+1}{2x}\right)^2$$

$$l = \int_1^2\sqrt{1+y'^2}\,dx = \int_1^2\frac{x^2+1}{2x}dx = \frac12\left(\int_1^2 x\,dx + \int_1^2\frac{dx}{x}\right)$$

$$= \frac12\left(\frac{x^2}{2}+\ln x\right)\Big|_1^2 = \frac34 + \frac12\ln 2$$

$$O = \frac34 + \cancel{\frac12\ln 2} + \frac14 + 1 - \cancel{\frac12\ln 2} + 1 = 3$$

III) Запремина и површина ротационе површи

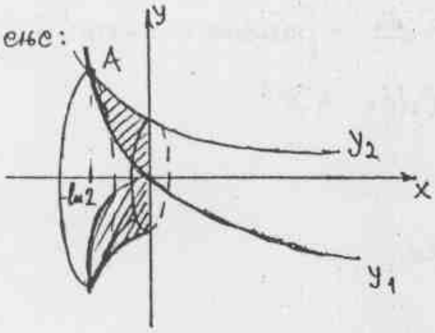


$$V = \pi \int_a^b f^2(x)\, dx$$

$$P = 2\pi \int_a^b |f(x)| \sqrt{1+(f'(x))^2}\, dx$$

Зад. 1. Израчунати запремину фигуре ограничене линијама $y_1 = e^{-2x}-1$, $y_2 = e^{-x}+1$, $x=0$ настале ротацијом око x-осе.

Решење:



$A(-\ln 2, 3)$

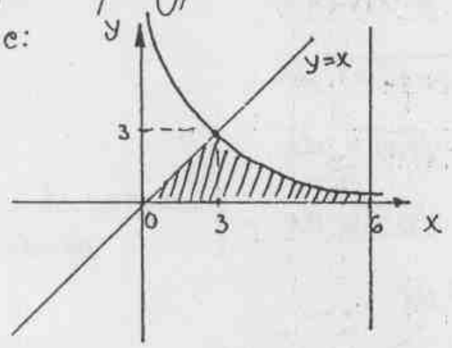
$$V = V_2 - V_1 = \pi \int_{-\ln 2}^{0} (e^{-x}+1)^2 dx - \pi \int_{-\ln 2}^{0} (e^{-2x}-1)^2 dx$$

$$= \pi \int_{-\ln 2}^{0} (3e^{-2x} - e^{-4x} + 2e^{-x}) dx$$

$$= \pi \left[-\tfrac{3}{2} e^{-2x} + \tfrac{1}{4} e^{-4x} + 2e^{-x}\right]_{\ln 2}^{0} = \cdots = \frac{11\pi}{4}$$

Зад. 2. Израчунати запремину ротационог тела насталог ротацијом дате фигуре око x-осе $xy=9$, $y=x$, $x=6$, $y=0$

Решење:



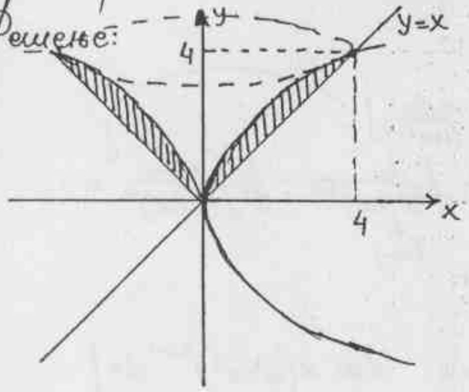
$$V_T = V_1 + V_2$$

$$= \pi \int_0^3 x^2 dx + \pi \int_3^6 \frac{81}{x^2} dx$$

$$= \cdots$$

Зад. 3. Крива ротира око Oy-осе и задата је фигура следећим кривама $y^2 = 4x$, $y = x$.

Решење:



$$V = \pi \int_a^b x^2(y)\, dy$$

$$x_1 = \frac{y^2}{4} \qquad x_2 = y$$

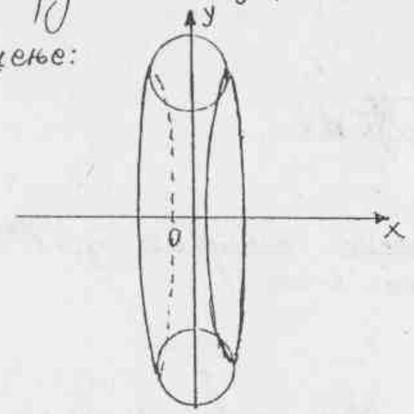
$$V = \pi \int_a^b x^2(y)\, dy$$

$$V_T = V_2 - V_1 = \pi \int_0^4 y^2 dy - \pi \int_0^4 \frac{y^4}{16} dy$$

$$= \cdots$$

Зад. 4. Израчунати запремину и површину тела насталог ротацијом круга $x^2+(y-2)^2=1$ око x-осе.

Решење:

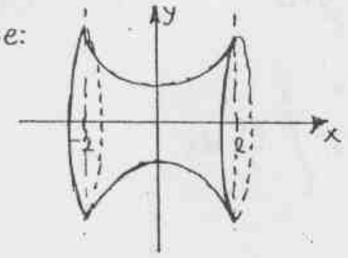


$$(y-2)^2 = 1-x^2$$
$$y-2 = \pm\sqrt{1-x^2} \qquad y_1 = 2+\sqrt{1-x^2}$$
$$y = 2\pm\sqrt{1-x^2} \qquad y_2 = 2-\sqrt{1-x^2}$$
$$V_T = V_1 - V_2 = \pi\int_{-1}^{1} y_1^2\,dx - \pi\int_{-1}^{1} y_2^2\,dx$$
$$= \pi\int_{-1}^{1}\left(\left(2+\sqrt{1-x^2}\right)^2 - \left(2-\sqrt{1-x^2}\right)^2\right)dx$$
$$= 8\pi\int_{-1}^{1}\sqrt{1-x^2}\,dx = [\text{смена: } x=\sin t]$$
$$= 8\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,dt = 4\pi^2$$

$$P = 2\pi\int_a^b |f(x)|\sqrt{1+(f'(x))^2}\,dx$$
$$P = P_1 + P_2 = 2\pi\int_{-1}^{1}\left(y_1\sqrt{1+y_1'^2}\,dx\right) + 2\pi\int_{-1}^{1} y_2\sqrt{1+y_2'^2}\,dx$$
$$= \dots = 8\pi\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} = 8\pi^2$$

Зад. 5. $y = \operatorname{ch} x \quad , \quad |x|\le 2 \quad , \quad P = ?$

Решење:



$$f(x) = \operatorname{ch} x \Rightarrow f'(x) = \operatorname{sh} x$$
$$P = 2\pi\int_{-2}^{2} f(x)\sqrt{1+f'^2(x)}\,dx$$
$$= 2\pi\int_{-2}^{2}\operatorname{ch} x\cdot\sqrt{1+\operatorname{sh}^2 x}\,dx$$
$$= 4\pi\int_0^2 \operatorname{ch} x\cdot\sqrt{1+\operatorname{sh}^2 x}\,dx = \left[\begin{array}{l}\text{смена: } \operatorname{sh} x = t\\ \operatorname{ch} x\,dx = dt\end{array}\right]$$
$$= 4\pi\int_0^{\operatorname{sh} 2}\sqrt{1+t^2}\,dt$$

I начин: $\operatorname{tg} x = t$

II начин: п. и.

III начин: $\int\frac{1+t^2}{\sqrt{1+t^2}}\,dt = (At+B)\sqrt{1+t^2} + \lambda\int\frac{dt}{\sqrt{1+t^2}}$

Зад. 6. $I_n = \int\frac{dx}{(x^2+a^2)^n}$

Решење: $\int\frac{dx}{(1+x^2)} = \int\frac{1+x^2-x^2}{(1+x^2)^2}\,dx = \int\frac{dx}{1+x^2} - \int\frac{x^2\,dx}{(1+x^2)^2}\left[\begin{array}{l}u=x\\ dv=\frac{x\,dx}{(1+x^2)^2}\end{array}\right]$

$$I_n = \int\frac{dx}{(x^2+a^2)^n} = \frac{1}{a^2}\int\frac{a^2}{(x^2+a^2)^n}\,dx = \frac{1}{a^2}\int\frac{a^2+x^2-x^2}{(x^2+a^2)^n}\,dx = \frac{1}{a^2}\underbrace{\int\frac{dx}{(x^2+a^2)^{n-1}}}_{I_{n-1}} - \frac{1}{a^2}\int\frac{x^2\,dx}{(x^2+a^2)^n} = \dots$$

Зад. 7. $I_n = \int(a^2-x^2)^n\,dx \qquad (a^2-x^2)^n = (a^2-x^2)(a^2-x^2)^{n-1}$

Решење: $I_n = a^2\underbrace{\int(a^2-x^2)^{n-1}\,dx}_{I_{n-1}} - \int x^2(a^2-x^2)^{n-1}\,dx \quad [u=x \quad dv = x(a^2-x^2)^{n-1}\,dx]$

Зад. 8. $I_n = \int_0^{\pi} x^n\sin x\,dx$

Решење: $I_n = \left[\begin{array}{ll}u=x^n & dv=\sin x\,dx\\ du=nx^{n-1}\,dx & v=-\cos x\end{array}\right] = -x^n\cos x\Big|_0^{\pi} + n\int_0^{\pi} x^{n-1}\cos x\,dx\left[\begin{array}{l}u=x^{n-1}\\ dv=\cos x\,dx\end{array}\right] =$

$$= \pi^n - n(n-1)I_{n-2}, \quad n\ge 2$$