

Dalamberov kriterijum

Ako za red $\sum_{n=1}^{\infty} a_n$ postoji $\overline{\lim}_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = r$ onda važi:

- za $r > 1$ red divergira
- za $r = 1$ neodlučivo
- za $r < 1$ konvergira

Primer 1.

Ispitati konvergenciju reda $\sum_{n=1}^{\infty} \frac{1}{n!}$

Rešenje:

Najpre da odredimo a_n . Ovde je to $a_n = \frac{1}{n!}$ (znači uzimamo sve iza oznake za red). Dalje odredjujemo a_{n+1} . Kako ?

Gledamo a_n i umesto n stavimo $n+1$, pa je $a_{n+1} = \frac{1}{(n+1)!}$

Sada koristimo Dalamberov kriterijum:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right| = \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = \lim_{n \rightarrow \infty} \frac{\cancel{n!}}{(n+1) \cdot \cancel{n!}} = \lim_{n \rightarrow \infty} \frac{1}{(n+1)} = 0$$

Dakle, dobili smo da je $r = 0 < 1$, pa po ovom kriterijumu , red $\sum_{n=1}^{\infty} \frac{1}{n!}$ konvergira.

Primer 2.

Ispitati konvergenciju reda $\sum_{n=1}^{\infty} \frac{2^n}{n}$

Rešenje:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{2^{n+1}}{n+1}}{\frac{2^n}{n}} \right| = \lim_{n \rightarrow \infty} \frac{n \cdot 2^{n+1}}{2^n \cdot (n+1)} = \lim_{n \rightarrow \infty} \frac{n \cdot \cancel{2^n} \cdot 2}{\cancel{2^n} \cdot (n+1)} = 2 \lim_{n \rightarrow \infty} \frac{n}{n+1} = 2 \cdot 1 = 2$$

Ovde smo dobili da je $r = 2$, a to nam govori da je red $\sum_{n=1}^{\infty} \frac{2^n}{n}$ divergentan.

Primer 3.

Ispitati konvergenciju reda $\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{2^{n+1}}$

Rešenje:

Ovde je $a_n = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{2^{n+1}}$, a da se podsetimo šta znači ovaj dvostruki faktorijel.

$$n! = n(n-1)(n-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1$$

$$n!! = n(n-2) \cdot (n-4) \cdot \dots$$

Zavisno da li je n paran ili neparan, kad ima !! stignemo do 2 ili 1. Recimo:

$$10!! = 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2$$

$$9!! = 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1$$

Ovde vam savetujemo da vodite računa o zagradama (recimo $(n!)! \neq n!!$)

Da se vratimo na zadatak:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{\frac{(2n+1)!!}{(2n+2)!!} \cdot \frac{1}{2^{n+2}}}{\frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{2^{n+1}}} = \lim_{n \rightarrow \infty} \frac{2^{n+1}}{2^{n+2}} \cdot \frac{(2n+1)!!}{(2n-1)!!} \cdot \frac{(2n)!!}{(2n+2)!!} = \\ &= \lim_{n \rightarrow \infty} \frac{\cancel{2^{n+1}}}{\cancel{2^{n+1}} \cdot 2} \cdot \frac{(2n+1) \cancel{(2n-1)!!}}{\cancel{(2n-1)!!}} \cdot \frac{\cancel{(2n)!!}}{(2n+2) \cancel{(2n)!!}} = \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} \cdot \frac{2n+1}{2n+2} = \boxed{\frac{1}{2}} \end{aligned}$$

Dakle, $r = \frac{1}{2}$, pa dati red konvergira.

Primer 4.

Ispitati konvergenciju reda $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Rešenje:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} = \lim_{n \rightarrow \infty} \frac{\cancel{(n+1)} \cdot \cancel{n!}}{\cancel{n!}} \cdot \frac{n^n}{\cancel{(n+1)} (n+1)^n} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n = \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{\frac{n+1}{n}} \right)^n = \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{1}{n}} \right)^n = \frac{1}{e} \end{aligned}$$

Ajde da resimo ovo i na drugi način:

Upotrebićemo trikče koje se često koristi kad imamo $n!$. To je takozvana Stirlingova aproksimacija:

$$n! \approx \sqrt{2n\pi} \cdot n^n \cdot e^{-n}$$

Sada imamo:

$$a_n = \frac{n!}{n^n} \sim \frac{\sqrt{2n\pi} \cdot \cancel{n^n} \cdot e^{-n}}{\cancel{n^n}} = \frac{\sqrt{2n\pi}}{e^n}$$

Probamo opet Dalamberov kriterijum:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{\frac{\sqrt{2(n+1)\pi}}{e^{n+1}}}{\frac{\sqrt{2n\pi}}{e^n}} = \lim_{n \rightarrow \infty} \frac{e^n}{e^{n+1}} \cdot \frac{\sqrt{2(n+1)\pi}}{\sqrt{2n\pi}} = \lim_{n \rightarrow \infty} \frac{\cancel{e^n} \cdot \sqrt{2\cancel{n}\pi(n+1)}}{\cancel{e^n} \cdot e \cdot \sqrt{2\cancel{n}\pi n}} = \\ &= \lim_{n \rightarrow \infty} \frac{1}{e} \cdot \sqrt{\frac{(n+1)}{n}} = \boxed{\frac{1}{e}} \end{aligned}$$

teži 1

Dakle $r = 1/e$, pa ovaj red konvergira...

Primer 5.

Ispitati konvergenciju reda $\sum_{n=1}^{\infty} \frac{n^p}{n!}$

Rešenje:

Ako probamo bez Stirlingove aproksimacije, imamo:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{\frac{(n+1)^{p+1}}{(n+1)!}}{\frac{n^p}{n!}} = \lim_{n \rightarrow \infty} \frac{(n+1)^{p+1}}{n^p} \cdot \frac{n!}{(n+1)!} = \lim_{n \rightarrow \infty} \frac{(n+1)^p \cdot \cancel{(n+1)}}{n^p} \cdot \frac{\cancel{n!}}{\cancel{(n+1)} \cdot \cancel{n!}} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^p = \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^p = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^{n \cdot \frac{p}{n}} = e^{\lim_{n \rightarrow \infty} \frac{p}{n}} = e^0 = 1 \end{aligned}$$

Kriterijum je neodlučiv, primenimo aproksimaciju:

$$a_n = \frac{n^p}{n!} \sim \frac{n^p}{\sqrt{2n\pi} \cdot n^n \cdot e^{-n}} = \frac{n^p \cdot e^n}{\sqrt{2n\pi} \cdot n^n}$$

Sad ćemo opet probati isti kriterijum:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{\frac{(n+1)^p \cdot e^{n+1}}{\sqrt{2(n+1)\pi} \cdot (n+1)^{n+1}}}{\frac{n^p \cdot e^n}{\sqrt{2n\pi} \cdot n^n}} = \lim_{n \rightarrow \infty} \frac{\sqrt{2n\pi}}{\sqrt{2(n+1)\pi}} \frac{e^{n+1}}{e^n} \frac{n^n}{(n+1)^{n+1}} \frac{(n+1)^p}{n^p} =$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{2n\pi}}{\sqrt{2(n+1)\pi}} \frac{e^n \cdot e}{e^n} \frac{n^n}{(n+1)^n \cdot (n+1)} \frac{(n+1)^p}{n^p} =$$

$$\lim_{n \rightarrow \infty} e \cdot \left(\frac{n}{n+1} \right)^n \frac{1}{n+1} \left(\frac{n+1}{n} \right)^p = \lim_{n \rightarrow \infty} e \cdot \left(1 + \frac{n}{n+1} - 1 \right)^n \frac{1}{n+1} = \lim_{n \rightarrow \infty} e \cdot \left(1 + \frac{n-n-1}{n+1} \right)^n \frac{1}{n+1}$$

$$= \lim_{n \rightarrow \infty} e \cdot \left(1 + \frac{-1}{n+1} \right)^n \frac{1}{n+1} = e \lim_{n \rightarrow \infty} \left(1 + \frac{1}{-n-1} \right)^{n \cdot (-n-1) \cdot \frac{1}{(-n-1)}} \frac{1}{n+1} =$$

$$= e \lim_{n \rightarrow \infty} \left(1 + \frac{1}{-n-1} \right)^{(-n-1) \cdot \frac{n}{(-n-1)}} \frac{1}{n+1} = e \cdot e^{-1} \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$$

Dakle, dati red konvergira!

Košijev koreni kriterijum:

Ako za red $\sum_{n=1}^{\infty} a_n$ postoji $\overline{\lim}_{n \rightarrow \infty} \sqrt[n]{|a_n|} = p$ onda važi :

- za $p > 1$ red divergira
- za $p = 1$ neodlučivo
- za $p < 1$ konvergira

Primer 6.

Ispitati konvergenciju reda $\sum_{n=1}^{\infty} \left(\frac{n-1}{n+1} \right)^{n(n-1)}$

Rešenje:

$$\begin{aligned}\lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{n-1}{n+1}\right)^{n(n-1)}} &= \lim_{n \rightarrow \infty} \left(\frac{n-1}{n+1}\right)^{\frac{n(n-1)}{n}} = \lim_{n \rightarrow \infty} \left(\frac{n-1}{n+1}\right)^{n-1} = \\ \lim_{n \rightarrow \infty} \left(1 + \frac{n-1}{n+1} - 1\right)^{n-1} &= \lim_{n \rightarrow \infty} \left(1 + \frac{n-1-n-1}{n+1}\right)^{n-1} = \lim_{n \rightarrow \infty} \left(1 + \frac{-2}{n+1}\right)^{n-1} = \\ &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{\frac{n+1}{-2}}\right)^{\frac{n+1}{-2} \cdot \frac{-2}{n+1} (n-1)} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{\frac{n+1}{-2}}\right)^{\frac{n+1}{-2} \cdot \frac{-2}{n+1} (n-1)} = e^{\lim_{n \rightarrow \infty} \frac{-2n+2}{n+1}} = e^{-2} = \boxed{\frac{1}{e^2}}\end{aligned}$$

Kako je $r = \frac{1}{e^2} < 1$, to znači da red $\sum_{n=1}^{\infty} \left(\frac{n-1}{n+1}\right)^{n(n-1)}$ konvergira po Košijevom kriterijumu.

Primer 7.

Ispitati konvergenciju reda $\sum_{n=1}^{\infty} \left(\frac{1+\cos n}{2+\cos n}\right)^{2n-\ln n}$

Rešenje:

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{1+\cos n}{2+\cos n}\right)^{2n-\ln n}} = \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{1+\cos n}{2+\cos n}\right)^{n\left(2-\frac{\ln n}{n}\right)}} = \lim_{n \rightarrow \infty} \left(\frac{1+\cos n}{2+\cos n}\right)^{2-\frac{\ln n}{n}}$$

Znamo da izraz $\frac{\ln n}{n}$ teži 0 kad n teži beskonačnosti, a da $\cos n$ ne može imati veću vrednost od 1. Onda je:

$$\lim_{n \rightarrow \infty} \left(\frac{1+\cos n}{2+\cos n}\right)^{2-\frac{\ln n}{n}} \leq \lim_{n \rightarrow \infty} \left(\frac{1+1}{2+1}\right)^{2-0} = \lim_{n \rightarrow \infty} \left(\frac{2}{3}\right)^2 = \frac{4}{9} < 1$$

Dakle, ovaj red konvergira.

Rabelov kriterijum:

Ako za red $\sum_{n=1}^{\infty} a_n$ postoji $\lim_{n \rightarrow \infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = t$ onda :

- za $t > 1$ konvergira
- za $t = 1$ neodlučiv
- za $t < 1$ divergira

Primer 8.

Ispitati konvergenciju reda $\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{2n+1}$

Rešenje:

$$\begin{aligned}
 \lim_{n \rightarrow \infty} n \left(\frac{a_n}{a_{n+1}} - 1 \right) &= \lim_{n \rightarrow \infty} n \left(\frac{\frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{2n+1}}{\frac{(2n+1)!!}{(2n+2)!!} \cdot \frac{1}{2n+3}} - 1 \right) = \lim_{n \rightarrow \infty} n \left(\frac{(2n-1)!!}{(2n+1)!!} \cdot \frac{(2n+2)!!}{(2n)!!} \cdot \frac{2n+3}{2n+1} - 1 \right) \\
 &= \lim_{n \rightarrow \infty} n \left(\frac{\cancel{(2n-1)!!}}{(2n+1) \cancel{(2n-1)!!}} \cdot \frac{(2n+2) \cancel{(2n)!!}}{\cancel{(2n)!!}} \cdot \frac{2n+3}{2n+1} - 1 \right) = \\
 &= \lim_{n \rightarrow \infty} n \left(\frac{(2n+2)(2n+3)}{(2n+1)^2} - 1 \right) = \lim_{n \rightarrow \infty} n \left(\frac{(2n+2)(2n+3) - (2n+1)^2}{(2n+1)^2} \right) = \\
 &= \lim_{n \rightarrow \infty} n \left(\frac{4n^2 + 6n + 4n + 6 - 4n^2 - 4n - 1}{(2n+1)^2} \right) = \lim_{n \rightarrow \infty} n \left(\frac{6n+5}{(2n+1)^2} \right) = \\
 &= \lim_{n \rightarrow \infty} \frac{6n^2 + 5n}{4n^2 + 4n + 1} = \frac{6}{4} = \boxed{\frac{3}{2} > 1}
 \end{aligned}$$

Znači da ovaj red, po Rabelovom kriterijumu konvergira.

Primer 9.

Nadji vrednost parametra p tako da red $\sum_{n=1}^{\infty} \frac{n!e^n}{n^{n+p}}$ konvergira.

Rešenje:

Najpre ćemo srediti izraz $\frac{a_n}{a_{n+1}}$

$$\begin{aligned}
 \frac{a_n}{a_{n+1}} &= \frac{\frac{n!e^n}{n^{n+p}}}{\frac{(n+1)!e^{n+1}}{(n+1)^{n+1+p}}} = \frac{n!}{(n+1)!} \cdot \frac{e^n}{e^{n+1}} \cdot \frac{(n+1)^{n+1+p}}{n^{n+p}} = \frac{n!}{(n+1) \cdot n!} \cdot \frac{e^n}{e^n \cdot e} \cdot \frac{(n+1)^{n+p} \cdot (n+1)}{n^{n+p}} = \frac{1}{e} \cdot \frac{(n+1)^{n+p}}{n^{n+p}} = \\
 &= \frac{1}{e} \left(\frac{n+1}{n} \right)^{n+p} = \frac{1}{e} \left(1 + \frac{1}{n} \right)^{n+p}
 \end{aligned}$$

Iskoristićemo trikče:

$$e^{\ln \Theta} = \Theta, \quad \text{gde je } \Theta = \left(1 + \frac{1}{n}\right)^{n+p}$$

$$\left(1 + \frac{1}{n}\right)^{n+p} = e^{\ln \left(1 + \frac{1}{n}\right)^{n+p}} = e^{(n+p) \ln \left(1 + \frac{1}{n}\right)}$$

Sada je

$$\frac{a_n}{a_{n+1}} = \frac{1}{e} \left(1 + \frac{1}{n}\right)^{n+p} = \frac{1}{e} e^{(n+p) \ln \left(1 + \frac{1}{n}\right)} = e^{-1} \cdot e^{(n+p) \ln \left(1 + \frac{1}{n}\right)} = e^{-1 + (n+p) \ln \left(1 + \frac{1}{n}\right)}$$

$$\ln \left(1 + \frac{1}{n}\right) \text{ moramo razviti koristeći: } \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad -1 < x < 1$$

$$\ln \left(1 + \frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right) \text{ sada je:}$$

$$\frac{a_n}{a_{n+1}} = e^{-1 + (n+p) \ln \left(1 + \frac{1}{n}\right)} = e^{-1 + (n+p) \left(\frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)\right)} = e^{-1 + 1 - \frac{1}{2n} + \frac{p}{n} - \frac{p}{2n^2} + o\left(\frac{1}{n^2}\right)} = e^{\frac{p}{n} - \frac{1}{2n} + o\left(\frac{1}{n}\right)} = e^{\frac{p - \frac{1}{2}}{n} + o\left(\frac{1}{n}\right)} = 1 + \frac{p - \frac{1}{2}}{n} + o\left(\frac{1}{n}\right) \text{ kad } n \rightarrow \infty$$

Dalje ćemo iskoristiti Rabelov kriterijum:

$$\lim_{n \rightarrow \infty} n \left(\frac{a_n}{a_{n+1}} - 1 \right) = \lim_{n \rightarrow \infty} n \left(1 + \frac{p - \frac{1}{2}}{n} - 1 \right) = \lim_{n \rightarrow \infty} n \frac{p - \frac{1}{2}}{n} = p - \frac{1}{2}$$

Sada, ako je:

$$p - \frac{1}{2} > 1 \rightarrow p > \frac{3}{2} \text{ red konvergira}$$

Košijev integralni kriterijum:

Ako funkcija $f(x)$ opada, neprekidna je i pozitivna, tada red $\sum_{n=1}^{\infty} f(n)$ konvergira ili divergira istovremeno sa

$$\text{integralom } \int_1^{\infty} f(x) dx$$

Primer 10.

Ispitati konvergenciju reda $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$

Rešenje:

Posmatramo integral: $\int_1^{\infty} \frac{1}{x^{\alpha}} dx$

$$\int_1^{\infty} \frac{1}{x^{\alpha}} dx = \lim_{A \rightarrow \infty} \int_1^A \frac{1}{x^{\alpha}} dx = \lim_{A \rightarrow \infty} \int_1^A x^{-\alpha} dx = \lim_{A \rightarrow \infty} \frac{x^{-\alpha+1}}{-\alpha+1} \Big|_1^A = \lim_{A \rightarrow \infty} \left(\frac{A^{-\alpha+1}}{-\alpha+1} - \frac{1}{-\alpha+1} \right) =$$

$$i) \text{ Ako je } \alpha > 1 \text{ onda je } \lim_{A \rightarrow \infty} \left(\frac{A^{-\alpha+1}}{-\alpha+1} - \frac{1}{-\alpha+1} \right) = 0 - \frac{1}{-\alpha+1} = \frac{1}{\alpha-1}$$

$$ii) \text{ Ako je } \alpha \leq 1 \text{ onda je } \lim_{A \rightarrow \infty} \left(\frac{A^{-\alpha+1}}{-\alpha+1} - \frac{1}{-\alpha+1} \right) = \infty$$

Dakle, red konvergira za $\alpha > 1$, a divergira za $\alpha \leq 1$.

Primer 11.

Ispitati konvergenciju reda sa opštim članom $a_n = \frac{1}{n \ln^p n}$ gde je $n > 1$

Rešenje:

$$\int_2^{\infty} \frac{1}{x \ln^p x} dx = \lim_{A \rightarrow \infty} \int_2^A \frac{1}{x \ln^p x} dx$$

Rešimo najpre integral na stranu bez granica (da ne bi morali da menjamo granice jer moramo upotrebiti smenu)

$$\int \frac{1}{x \ln^p x} dx = \left| \begin{array}{l} \ln x = t \\ \frac{1}{x} dx = dt \end{array} \right| = \int \frac{1}{t^p} dt = \int t^{-p} dt = \frac{t^{-p+1}}{-p+1} = \frac{t^{1-p}}{1-p}$$

$$\int_2^{\infty} \frac{1}{x \ln^p x} dx = \lim_{A \rightarrow \infty} \int_2^A \frac{1}{x \ln^p x} dx = \lim_{A \rightarrow \infty} \frac{(\ln x)^{1-p}}{1-p} \Big|_2^A = \lim_{A \rightarrow \infty} \frac{(\ln A)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p} =$$

$$i) \text{ Ako je } 1-p < 0 \rightarrow p > 1 \text{ konvergira } \frac{(\ln A)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p} = 0 - \frac{(\ln 2)^{1-p}}{1-p} = \frac{(\ln 2)^{1-p}}{p-1}$$

$$ii) \text{ Ako je } p < 1 \text{ divergira } \lim_{A \rightarrow \infty} \frac{(\ln A)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p} = \infty$$

Gausov kriterijum: Ako za red $\sum_{n=1}^{\infty} a_n$ sa pozitivnim članovima postoji:

$$\frac{a_n}{a_{n+1}} = \lambda + \frac{\mu}{n} + o\left(\frac{1}{n^{1+\varepsilon}}\right) \quad \text{za} \quad \forall \varepsilon > 0 \quad \text{tada:}$$

- i) Ako je $\lambda > 1$ red konvergira
- ii) Ako je $\lambda < 1$ red divergira
- iii) Ako je $\lambda = 1$ tada $\begin{cases} \text{za } \mu > 1 \text{ red konvergira} \\ \text{za } \mu < 1 \text{ red divergira} \end{cases}$

Primer 12.

Ispitati konvergenciju reda $\sum_{n=1}^{\infty} \left[\frac{(2n-1)!!}{(2n)!!} \right]^p$

Rešenje:

$$\frac{a_n}{a_{n+1}} = \frac{\left[\frac{(2n-1)!!}{(2n)!!} \right]^p}{\left[\frac{(2n+1)!!}{(2n+2)!!} \right]^p} = \left[\frac{(2n-1)!!}{(2n+1)!!} \cdot \frac{(2n+2)!!}{(2n)!!} \right]^p = \left[\frac{(2n-1)!!}{(2n+1)(2n-1)!!} \cdot \frac{(2n+2)(2n)!!}{(2n)!!} \right]^p = \left[\frac{2n+2}{2n+1} \right]^p$$

Sad spakujemo malo ovaj izraz i upotrebljavamo binomnu formulu:

$$\begin{aligned} \left[\frac{2n+2}{2n+1} \right]^p &= \left[\frac{2n+1+1}{2n+1} \right]^p = \left[1 + \frac{1}{2n+1} \right]^p = \\ &= \binom{p}{0} 1^p \left(\frac{1}{2n+1} \right)^0 + \binom{p}{1} 1^{p-1} \left(\frac{1}{2n+1} \right)^1 + \binom{p}{2} 1^{p-2} \left(\frac{1}{2n+1} \right)^2 + \dots \\ &= 1 + \frac{p}{2n+1} + \underbrace{\left| \frac{p(p+1)}{2(2n+1)^2} + o\left(\frac{1}{n^2}\right) \right|} \\ &= 1 + \frac{p}{2n+1} + o\left(\frac{1}{n^2}\right) \\ &= 1 + \frac{p}{2\left(n + \frac{1}{2}\right)} + o\left(\frac{1}{n^2}\right) \\ &= 1 + \frac{p/2}{n+1/2} + o\left(\frac{1}{n^2}\right) \quad \text{kad } n \rightarrow \infty \\ &= 1 + \frac{p/2}{n} + o\left(\frac{1}{n^2}\right) \end{aligned}$$

Ovo upoređujemo sa $\frac{a_n}{a_{n+1}} = \lambda + \frac{\mu}{n} + o(\frac{1}{n^{1+\varepsilon}})$

Jasno je da je $\lambda=1$ pa nam treba $\mu = \frac{p}{2}$

i) Ako je $\mu = \frac{p}{2} > 1 \rightarrow p > 2$ red konvergira

ii) Ako je $\mu = \frac{p}{2} < 1 \rightarrow p < 2$ red divergira

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