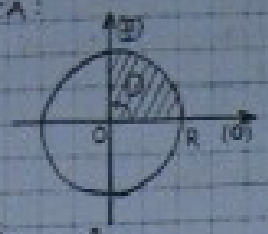
пример 2: $I=\iint\limits_D xy\,dx\,dy$, $D: x^2+y^2\le R^2$, $x,y\ge 0$

СЛИКА:



СМЕНА:

$x=\rho\cdot\cos\varphi$
$y=\rho\cdot\sin\varphi$

$$I=\int_0^{\pi/2}d\varphi\cdot\int_0^R \rho\cos\varphi\cdot\rho\sin\varphi\cdot\underbrace{\rho\,d\rho}_{|J|}=\int_0^{\pi/2}d\varphi\cdot I_1$$

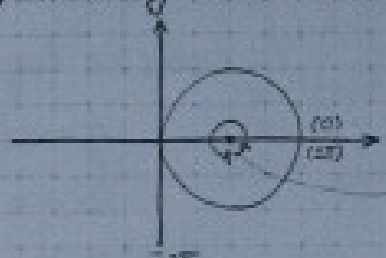
$$I_1=\int_0^R\rho^3\cos\varphi\sin\varphi\,d\rho=\sin\varphi\cos\varphi\cdot\int_0^R\rho^3\,d\rho=\sin\varphi\cos\varphi\,\frac{\rho^4}{4}\Big|_0^R=\frac{1}{4}\sin\varphi\cos\varphi\cdot R^4$$

$$I=\frac{R^4}{4}\int_0^{\pi/2}\sin\varphi\cos\varphi\,d\varphi=\frac{R^4}{8}\int_0^{\pi/2}\sin 2\varphi\,d\varphi=\frac{R^4}{16}\cos 2\varphi=\frac{R^4}{16}+\frac{R^4}{16}=\frac{R^4}{8}\Rightarrow\boxed{I=\frac{R^4}{8}}$$

пример 3: $I=\iint\limits_D (x+y)\,dx\,dy$, $D: x^2+y^2\le 2x$

$D: x^2-2x+y^2\le 0$

$(x-1)^2+y^2\le 1$



$R=1 \Rightarrow 0\le\rho\le 1$

$\alpha=0,\ \beta=2\pi$

СМЕНА: $x=1+\rho\cos\varphi$
$y=\rho\sin\varphi$

$$I=\int_0^{2\pi}d\varphi\cdot\int_0^1(1+\rho\cos\varphi+\rho\sin\varphi)\cdot\rho\cdot d\rho=\int_0^{2\pi}d\varphi\cdot I_1$$

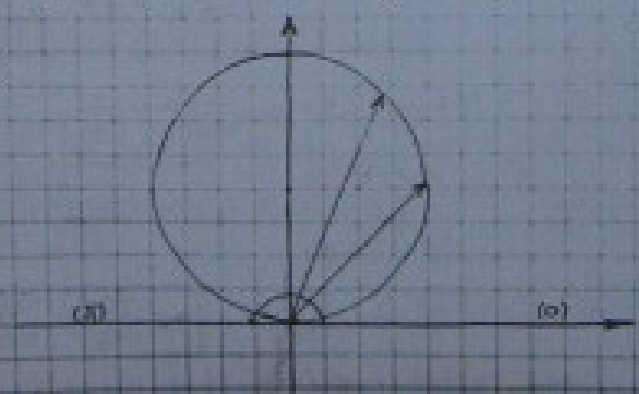
$$I_1=\int_0^1(\rho+\rho^2\cdot(\sin\varphi+\cos\varphi))\,d\rho=\int_0^1\rho\,d\rho+(\sin\varphi+\cos\varphi)\int_0^1\rho^2\,d\rho=$$

$$=\frac{\rho^2}{2}\Big|_0^1+(\sin\varphi+\cos\varphi)\frac{\rho^3}{3}\Big|_0^1=\frac{1}{2}+\frac{1}{3}(\sin\varphi+\cos\varphi)$$

$$I=\int_0^{2\pi}\left(\frac{1}{2}+\frac{1}{3}(\sin\varphi+\cos\varphi)\right)d\varphi=\frac{1}{2}\int_0^{2\pi}d\varphi+\frac{1}{3}\int_0^{2\pi}\sin\varphi\,d\varphi+\frac{1}{3}\int_0^{2\pi}\cos\varphi\,d\varphi=$$

$$=\frac{1}{2}\varphi\Big|_0^{2\pi}-\frac{1}{3}\cos\varphi\Big|_0^{2\pi}+\frac{1}{3}\sin\varphi\Big|_0^{2\pi}=\pi-\frac{1}{3}+\frac{1}{3}+0=\pi\Rightarrow\boxed{I=\pi}$$

пример 4: $I=\iint\limits_D\sqrt{x^2+y^2}\,dx\,dy$, $D: x^2+y^2\le 4y$

$D: x^2+y^2-4y\le 0$

$x^2+(y-2)^2\le 4$